# THE INTEGRAL HOMOLOGY OF $\mathrm{PSL}_2$ OF IMAGINARY QUADRATIC INTEGERS WITH NON-TRIVIAL CLASS GROUP

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Abstract. We show that a cellular complex described by Flöge allows to determine the integral homology of the *Bianchi groups* $\mathrm{PSL}_2(\mathcal{O}_{-m})$, where $\mathcal{O}_{-m}$ is the ring of integers of an imaginary quadratic number field $\mathbb{Q}\left[\sqrt{-m}\right]$ for a square-free natural number $m$. We use this to compute in the cases $m = 5, 6, 10, 13$ and 15 with non-trivial class group the integral homology of $\mathrm{PSL}_2(\mathcal{O}_{-m})$, which before was known only in the cases $m = 1, 2, 3, 7$ and 11 with trivial class group.

Contents



## 1. Introduction

The object of study of this paper are the $\mathrm{PSL}_2$-groups $\Gamma$ of the ring of integers $\mathcal{O}_{-m} := \mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}$ of an imaginary quadratic number field $\mathbb{Q}[\sqrt{-m}\,]$, where $m$ is a square-free positive integer. We have $\mathcal{O}_{-m} = \mathbb{Z}[\omega]$ with $\omega = \sqrt{-m}$ for $m$ congruent to 1 or 2 modulo 4, and $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{-m}$ for $m$ congruent to 3 modulo 4. The arithmetic groups under study have often been called Bianchi groups, because Luigi Bianchi [4] computed fundamental domains for them as early as in 1892. They act on $\mathrm{PSL}_2(\mathbb{C})$'s symmetric space, the hyperbolic three-space $\mathcal{H}$. This action rose interest when Felix Klein and Henri Poincaré studied certain groups of Möbius transformations with complex coefficients [14], laying the ground for the concept of *Kleinian groups*. Each non-cocompact arithmetic Kleinian group is commensurable with some Bianchi group [11]. Thus, the Bianchi groups play a key role in the study of arithmetic Kleinian groups. A wealth of information on the Bianchi groups can be found in the pertinent monographs [6, 7, 11].

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Poincaré gave an explicit formula for their action on $\mathcal{H}$. However, the virtual cohomological dimension of arithmetic groups which are lattices in $\mathrm{SL}_2(\mathbb{C})$ is two, so it is desirable to restrain this proper action on $\mathcal{H}$ to a contractible cellular two-dimensional space. Moreover, this space should be cofinite. In principle, this has been achieved by Mendoza [12] and also by Flöge [8], using Minkowski's reduction theory. Their two approaches have in common that they consider two-dimensional $\Gamma$-equivariant retracts which are cocompact and are endowed with a natural CW-structure such that the action of $\Gamma$ is cellular and the quotient is a finite CW-complex.
Using Mendoza's complex, Schwermer and Vogtmann [15] calculated the integral group homology in the cases of trivial class group $m = 1, 2, 3, 7, 11$, and Vogtmann [19] computed the rational homology as the homology of the quotient space in many cases of non-trivial class group. The integral cohomology in the cases $m = 2, 3, 5, 6, 7, 10, 11$ has been determined by Berkove [3], based on Flöge's presentation of the groups with generators and relations.
It is the purpose of the present paper to show how Flöge's complex can be used to obtain the integral homology of Bianchi groups also when the class group is non-trivial. Our results are as follows:

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-5});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geqslant 3; \end{cases}$$

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-10});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geqslant 3; \end{cases}$$

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-15});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2, \\ \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q \geqslant 3; \end{cases}$$

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-13});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2, & q = 2, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k+3, \quad k \geqslant 0, \\ (\mathbb{Z}/2)^q, & q = 4k+4, \quad k \geqslant 0, \\ (\mathbb{Z}/2)^q, & q = 4k+1, \quad k \geqslant 1, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k+2, \quad k \geqslant 1; \end{cases}$$

$$\mathrm{H}_q(\mathrm{PSL}_2(\mathcal{O}_{-6});\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k+3, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k+4, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+4}, & q = 6k+5, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k+6, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k}, & q = 6k+1,\ q \geqslant 7, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+3}, & q = 6k+2,\ q \geqslant 8. \end{cases}$$

Thus for $q \geqslant 2$, the torsion in $\mathrm{H}_*(\mathrm{PSL}_2(\mathcal{O}_{-5});\mathbb{Z})$ is the same as in $\mathrm{H}_*(\mathrm{PSL}_2(\mathcal{O}_{-10});\mathbb{Z})$, analogously to the cohomology results of Berkove [3]. The free part of these homology groups is in accordance with the rational homology results of Vogtmann [19].
In the cases of non-trivial ideal class group, there is a difference between the approaches of Mendoza

and Flöge. We use the upper-half-space model of $\mathcal{H}$ and identify its boundary with $\mathbb{C} \cup \infty \cong \mathbb{C}P^1$. The elements of the class group of the number field are in bijection with the $\Gamma$-orbits of the cusps, where the cusps are $\infty$ and the elements of the number field $\mathbb{Q}\left[\sqrt{-d}\,\right]$, thought of as elements of the canonical boundary $\mathbb{C}P^1$. The cusps which represent a non-trivial element of the class group are commonly called *singular* points. Whilst Mendoza retracts away from all cusps, Flöge retracts away only from the non-singular ones. Rather than the space $\mathcal{H}$ itself, he considers the space $\widehat{\mathcal{H}}$ obtained from $\mathcal{H}$ by adjoining the $\Gamma$-orbits of the singular points. Then, the geodesic retraction of $\mathcal{H}$ extends naturally to one of $\widehat{\mathcal{H}}$, including the singular cusps into the retract $X$ of $\widehat{\mathcal{H}}$. Now it turns out that the quotient space by $\Gamma$ of $X$ is compact, and $X$ is a suitable contractible 2-dimensional $\Gamma$-complex also in the case of non-trivial class group.

With an implementation in Pari/GP [2], due to the first named author, of Swan's algorithm [18] we obtain a fundamental polyhedron for $\Gamma$ in $\mathcal{H}$. In the cases considered, Bianchi has already computed this polyhedron, so we have a control of the correctness of the implementation.

In the cases $m = 5, 6$ and 10, Flöge has computed the cell stabilizers and cell identifications; and with our Pari/GP program, we redo Flöge's computations and do the same computation in the cases $m = 13$ and 15. We use the equivariant Euler characteristic to check our computations. Then we follow the lines of Schwermer and Vogtmann [15], encountering a spectral sequence which degenerates on the $E^3$-page and not already on the $E^2$-page as it does in the cases of trivial class group. This is because of the singular points in our cell complex $X$, which have infinite stabilizers. So we have some additional use of homological algebra to obtain the homology of the Bianchi group. We give the full details for our homology computation in the case $m = 13$. We then give slightly less details in the cases $m = 5, 6, 10$ and 15.

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## 2. Flöge's complex, contractibility and the spectral sequence

Denote the hyperbolic three-space by $\mathcal{H} \cong \mathbb{C} \times \mathbb{R}^*_+$. We will not use its smooth structure, only its structure as a homogeneous $\mathrm{SL}_2(\mathbb{C})$-space. The action is given by the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (z, r) := \left( \frac{(\overline{d} - \overline{cz})(az - b) - r^2 \overline{c} a}{|cz - d|^2 + r^2 |c|^2}, \frac{r}{|cz - d|^2 + r^2 |c|^2} \right);$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$. As usual, we extend the action of $\mathrm{SL}_2(\mathbb{C})$ to the boundary $\mathbb{C}P^1$ which we identify with $\{r = 0\} \cup \infty \cong \mathbb{C} \cup \infty$. The action passes continuously to the boundary, where it reduces to the usual action by Möbius transformations $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot z = \frac{az - b}{-cz + d}$. As $-1 \in \mathrm{SL}_2(\mathbb{C})$ acts trivially, the action passes to $\mathrm{PSL}_2(\mathbb{C})$. Now, fix $m \in \mathbb{N}$ be square-free, $\mathcal{O}_{-m}$ the ring of integers in $\mathbb{Q}[\sqrt{-m}\,]$, and $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$. When the class number of $\mathbb{Q}[\sqrt{-m}\,]$ is one, then classical reduction theory provides a natural equivariant deformation retract of $\mathcal{H}$ which is a CW-complex. This complex is defined as follows. One first considers the union of all hemispheres

$$S_{\mu,\lambda} := \left\{ (z, r) : \quad \left| z - \frac{\lambda}{\mu} \right|^2 + r^2 = \frac{1}{|\mu|^2} \right\} \subset \mathcal{H},$$

for any two $\mu, \lambda$ with $\mu\mathcal{O}_{-m} + \lambda\mathcal{O}_{-m} = \mathcal{O}_{-m}$. Then one considers the "space above the hemispheres"

$$B := \left\{ (z, r) : \quad |cz - d|^2 + r^2 |c|^2 \geqslant 1 \text{ for all } c, d \in \mathcal{O}_{-m}, c \neq 0 \text{ such that } c\mathcal{O}_{-m} + d\mathcal{O}_{-m} = \mathcal{O}_{-m} \right\}$$

and its boundary $\partial B$ inside $\mathcal{H}$. For nontrivial class group, the following definition comes to work.

**Definition 1.** *A point $s \in \mathbb{C}P^1 - \{\infty\}$ is called a* singular point *if for all $c, d \in \mathcal{O}_{-m}$, $c \neq 0$, $c\mathcal{O}_{-m} + d\mathcal{O}_{-m} = \mathcal{O}_{-m}$ we have $|cs - d| \geqslant 1$.*

The singular points modulo the action of $\Gamma$ on $\mathbb{C}P^1$ are in bijection with the nontrivial elements of the class group [17]. In [8], Flöge extends the hyperbolic space $\mathcal{H}$ to a larger space $\widehat{\mathcal{H}}$ as follows.

**Definition 2.** *As a set, $\widehat{\mathcal{H}} \subset \mathbb{C} \times \mathbb{R}^{\geqslant 0}$ is the closure under the $\Gamma$-action, of the union $\widehat{B} := B \cup \{$singular points$\}$. The topology is generated by the topology of $\mathcal{H}$ together with the following neighborhoods of the translates $s$ of singular points:*

$$\widehat{U}_\epsilon(s) := \{s\} \cup \begin{pmatrix} s & 0 \\ -1 & s^{-1} \end{pmatrix} \cdot \left\{(z,r) \in \mathcal{H} : r > \epsilon^{-1}\right\}.$$

***Remark* 3.** The matrix $\left(\begin{smallmatrix} s & 0 \\ -1 & s^{-1} \end{smallmatrix}\right)$ maps the point at infinity into $s$, thus giving the point $s$ the topology of $\infty$. The neighborhood $\widehat{U}_\epsilon(s)$ is sometimes called a "horoball" because in the upper-half space model it is a Euclidean ball, but with the hyperbolic metric it has "infinite radius".

The space $\widehat{\mathcal{H}}$ is endowed with the natural $\Gamma$-action. Now the essential aspect of Flöge's construction is the following consequence of Flöge's theorem [9, 6.6], which we append as theorem 24.

**Corollary 4.** *There is a retraction $\rho$ from $\widehat{\mathcal{H}}$ onto the set $X \subset \widehat{\mathcal{H}}$ of all $\Gamma$-translates of $\partial\widehat{B}$, i. e. there is a continuous map $\rho : \widehat{\mathcal{H}} \to X$ such that $\rho(p) = p$ for all $p \in X$. The set $X$ admits a natural structure as a cellular complex $X^\bullet$, such that $\Gamma$ acts cellularly on $X^\bullet$.*

***Remark* 5.**
1. We show with the below lemma that $\rho$ is a homotopy equivalence, without giving a continuous path of maps $\widehat{\mathcal{H}} \to \widehat{\mathcal{H}}$ connecting $\rho$ to the identity on $\widehat{\mathcal{H}}$.
2. The map $\rho$ is $\Gamma$-equivariant because its fibers are geodesics. But we do not make use of this fact, as we do not need to show that the homotopy type of $\Gamma\backslash\widehat{\mathcal{H}}$ is the same as that of $\Gamma\backslash X$. This would be useful in the case of trivial class group, i. e. the case of a proper action, to compute the rational homology $\mathrm{H}_*(\Gamma; \mathbb{Q}) \cong \mathrm{H}_*(\Gamma\backslash\mathcal{H}; \mathbb{Q})$.
3. We will provide $X^\bullet$ with a cellular structure which is fine enough to make the cell stabilizers fix the cells pointwise.

**Lemma 6.** *Let $Y$ be a CW-complex which admits an inclusion $i$ into a contractible topological space $A$, such that $i$ is a homeomorphism between $Y$ with its cellular topology and the image $i(Y)$ with the subset topology of $A$. Let $p : A \to Y$ be a continuous map with $p \circ i = \mathrm{id}_Y$. Then $p$ is a homotopy equivalence.*

*Proof.* For all $n \in \mathbb{N}$, the induced maps on the homotopy groups $(\mathrm{id}_Y)_* = (p \circ i)_* : \pi_n(Y) \to \pi_n(Y)$ factor through $\pi_n(A) = 0$, hence are the zero map; and $\pi_n(Y) = 0$. Denote by $c$ the constant map from $A$ to the one-point space. Then $c \circ i$ is a morphism of CW-complexes, and the zero maps it induces on the homotopy groups are isomorphisms. Thus by Whitehead's Theorem, $c \circ i$ is a homotopy equivalence. As $A$ is contractible, the composition $(c \circ i) \circ p = c$ is a homotopy equivalence, so the same holds already for $p$. □

Taking $Y = X$, $A = \widehat{\mathcal{H}}$, $p = \rho$, and using lemma 8, we obtain a crucial fact for our computations.

**Corollary 7.** *$X^\bullet$ is contractible.*

The following is an observation on Flöge's construction.

**Lemma 8.** *The space $\widehat{\mathcal{H}}$ is contractible.*

*Proof.* One can identify the boundary of $\mathcal{H} \cong \{(z,r) \in \mathbb{C} \times \mathbb{R} \mid r > 0\}$ with $\mathbb{C}P^1 \cong \mathbb{C} \cup \infty \cong \{r = 0\} \cup \infty$. By viewing the singular points as part of the boundary, we arrive at an upper half-space model of $\widehat{\mathcal{H}}$. Now consider $\mathcal{H}_1 := \{(z,r) \in \widehat{\mathcal{H}} : r \geqslant 1\}$ with the subspace topology of $\widehat{\mathcal{H}}$. The idea of the proof is to

consider a vertical retraction onto $\mathcal{H}_1$, and to show by an explicit argument that preimages of open sets are open. Flöge [9, Korollar 5.8] suggests to use the map

$$\phi : \widehat{\mathcal{H}} \times [0,1] \to \widehat{\mathcal{H}},\ ((z,r),t) \mapsto \begin{cases} (z,r) \text{ for all } t \in [0,1], & \text{if } r \geqslant 1 \\ (z, r+t(1-r)), & \text{if } r < 1. \end{cases}$$

Let us now check that this is a continuous family of continuous maps. Consider the collection of open balls with respect to the Euclidean metric on $\mathbb{C} \times \mathbb{R}_+$ as soon as they are either contained in $\mathbb{C} \times \mathbb{R}_+^*$, or touch the boundary $\mathbb{C} \times \{0\}$ in a cusp in $\widehat{\mathcal{H}} - \mathcal{H}$. This is a basis for the topology of $\widehat{\mathcal{H}}$. Consider one such open ball $\mathcal{B}$, and its preimage under some $\phi_t$, $t \in [0,1)$. This either lies entirely in $\mathcal{H}$, and is open, or it has boundary points. In the latter case, consider the inverse of $\phi_t$ on $\widehat{\mathcal{H}} - \mathcal{H}_1$, given by

$$\phi_t^{-1} = \left(z, \tfrac{r-t}{1-t}\right),$$

if this is in $\widehat{\mathcal{H}}$. Suppose there is a cusp $s$ with $s \in \widehat{\mathcal{H}} - \mathcal{H}$ and $\phi_t(s,0) \in \mathcal{B}$. As $\mathcal{B}$ is open, we find $\beta > 0$ and $\delta > 0$ such that $(s, t+\beta)$ and $(s+\delta, t)$ are in $\mathcal{B}$. Since

$$\begin{cases} \phi_t\left(s, \frac{\beta}{1-t}\right) = (s, t+\beta) \in \mathcal{B} \\ \phi_t(s+\delta, 0) = (s+\delta, t) \in \mathcal{B}, \end{cases}$$

we know that $(s, \frac{\beta}{1-t})$ and $(s+\delta, 0)$ are in the preimage of $\mathcal{B}$ under $\phi_t$. We deduce that the whole horosphere of Euclidean diameter $\min\{\beta, \delta\}$ touching at the cusp $s$ is included in the preimage of $\mathcal{B}$. Thus each point of the preimage has a neighborhood entirely contained in the preimage, which therefore also is open. The continuity at $t = 1$ as well as the continuity in the variable $t$ follow from very similar arguments. The space $\mathcal{H}_1$ is homeomorphic to $\mathbb{C} \times \mathbb{R}_+$, thus contractible. □

2.0.1. *The spectral sequence.*
Corollary 7 gives us a contractible complex $X^\bullet$ on which $\Gamma$ acts cellularly. As a consequence, the integral homology $\mathrm{H}_*(\Gamma; \mathbb{Z})$ can be computed as the hyperhomology $\mathbb{H}_*(\Gamma; C_\bullet(X))$ of $\Gamma$ with coefficients in the cellular chain complex associated to $X$. This hyperhomology is computable because there is a spectral sequence as in [5, VII] which is also the one used in [15]. It is the spectral sequence associated to the double complex $\Theta_\bullet^\Gamma \otimes_{\mathbb{Z}\Gamma} C_\bullet(X)$ computing the hyperhomology, where we denote by $\Theta_\bullet^\Gamma$ the bar resolution of the group $\Gamma$. This spectral sequence can be rewritten (see [15, 1.1]) to yield

$$E^1_{p,q} = \bigoplus_{\sigma \in \Gamma \backslash X^p} \mathrm{H}_q(\Gamma_\sigma;\, \mathbb{Z}) \implies \mathrm{H}_{p+q}(\Gamma;\, \mathbb{Z}),$$

where $\Gamma_\sigma$ denotes the stabilizer of (the chosen representative for) the $p$-cell $\sigma$. We have stated the above $E^1$-term with trivial $\mathbb{Z}$-coefficients in $\mathrm{H}_q(\Gamma_\sigma; \mathbb{Z})$, because we use a fundamental domain which is strict enough to give $X$ a cell structure on which $\Gamma$ acts without inversion of cells. We shall also make extensive use of the description of the $d^1$-differential given in [15].
The technical difference to the cases of trivial class group, treated by [15], is that the stabilizers of the singular points are free abelian groups of rank two. In particular, the $\Gamma$-action on our complex $X^\bullet$ is not a *proper action* in the sense that all stabilizers would be finite. As a consequence, the considered spectral sequence does not degenerate on the $E^2$-level as it does in Schwermer and Vogtmann's cases. So we compute a nontrivial differential $d^2$, making some additional use of homological algebra, in particular the below lemma and its corollary.

***Remark* 9.** It would be possible to shift the technical difficulty away from homological algebra, using a topological modification of our complex. In our cases of class number two, there is one singular point in the fundamental domain, representing the nontrivial element of the class group. Its stabilizer is free abelian of rank two, and contributes the homology of a torus to the zeroth column of the $E^2$-term

of our spectral sequence: $\mathrm{H}_1(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}^2$, $\mathrm{H}_2(\mathbb{Z}^2;\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{H}_q(\mathbb{Z}^2;\mathbb{Z}) = 0$ for $q > 2$. One could modify our complex in order to make the $\Gamma$-action on it proper, by replacing each singular point by an $\mathbb{R}^2$ with the former stabilizer $\mathbb{Z}^2$ now acting properly. Then the nontriviality of our differential is equivalent to the existence of a nontrivial homology relation induced by adjoining the torus $\mathbb{R}^2/\mathbb{Z}^2$ to the fundamental domain.

The following lemma will be useful for computing our $d^2$-differential. In order to state it, let $\Gamma_\sigma$ be a finite subgroup of $\Gamma$, let $M$ be a $\mathbb{Z}\Gamma_\sigma$-module, and $\ell : \Gamma/\Gamma_\sigma \to \Gamma$ a set-theoretical section of the quotient map $\pi : \Gamma \to \Gamma/\Gamma_\sigma$. Further, denote the standard bar resolution of a discrete group $\Gamma$ by $\Theta^\Gamma_\bullet$.

**Lemma 10.** *The section $\ell$ defines a map of $\mathbb{Z}\Gamma_\sigma$-complexes*

$$\hat{\varepsilon_\ell} : \Theta^\Gamma_\bullet \longrightarrow \Theta^{\Gamma_\sigma}_\bullet$$

*of degree zero which is a retraction of the resolution $\Theta^\Gamma_\bullet$ of the group $\Gamma$ to the resolution $\Theta^{\Gamma_\sigma}_\bullet$ of $\Gamma_\sigma$. The map $\hat{\varepsilon_\ell}$ is induced on $\Theta^\Gamma_0 = \mathbb{Z}\Gamma$ by*

$$\Gamma \xrightarrow{\varepsilon_\ell} \mathbb{Z}\Gamma_\sigma,$$
$$\gamma \mapsto (\ell(\pi(\gamma)))^{-1}\gamma$$

*and is continued as a tensor product $\hat{\varepsilon_\ell} = \varepsilon_\ell \otimes \ldots \otimes \varepsilon_\ell = \varepsilon_\ell^{\otimes(n+1)}$ on $\Theta^\Gamma_n$.*

***Remark* 11.**
1. Attention: $\varepsilon_\ell$ is a $\mathbb{Z}\Gamma_\sigma$-linear map because $\Gamma_\sigma$ acts *from the right.*
2. Note that the resulting isomorphism in homology from $\mathrm{H}_*(\Theta^\Gamma_\bullet \otimes_{\mathbb{Z}\Gamma_\sigma} M)$ to $\mathrm{H}_*(\Theta^{\Gamma_\sigma}_\bullet \otimes_{\mathbb{Z}\Gamma_\sigma} M)$ is independent of the choice of $\ell$, and consistent with the canonical isomorphisms of both sides with $\mathrm{H}_*(\Gamma_\sigma; M)$.
3. Note that in the above lemma, it is not necessary to require $\ell(\pi(1)) = 1$. This would imply that $\varepsilon_\ell$ is the identity on $\Theta^{\Gamma_\sigma}_\bullet$. However, we will choose $\ell(\pi(1)) = 1$ for simplicity.
4. In explicit terms, the map $\varepsilon_\ell$ is described as follows:

$$\varepsilon_\ell : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma_\sigma,$$

$$\sum_{\gamma\in\Gamma} a_\gamma \gamma = \sum_{\gamma_\sigma\in\Gamma_\sigma}\sum_{\rho\in\Gamma/\Gamma_\sigma} a_{\gamma_\sigma\ell(\rho)}\gamma_\sigma\ell(\rho) \mapsto \sum_{\gamma_\sigma\in\Gamma_\sigma}\Big(\sum_{\rho\in\Gamma/\Gamma_\sigma} a_{\gamma_\sigma\ell(\rho)}\Big)\gamma_\sigma,$$

   where the $a_\gamma$ are coefficients from $\mathbb{Z}$. The map $\varepsilon_\ell$ restricts to the identity on $\mathbb{Z}\Gamma_\sigma$ and gives an isomorphism of $\mathbb{Z}$-modules from $\mathbb{Z}[\ell(\rho)\Gamma_\sigma]$ to $\mathbb{Z}\Gamma_\sigma$ for every $\Gamma_\sigma$-orbit $\ell(\rho)\Gamma_\sigma$.

*Proof (of the lemma).* In fact, the statement holds for any chain map $\hat{\varepsilon}$ in the place of $\hat{\varepsilon_\ell}$ that satisfies the following conditions. They are easily checked to hold for the maps $\hat{\varepsilon_\ell}$.

1. $\hat{\varepsilon}$ is $\mathbb{Z}\Gamma_\sigma$-linear.
2. The augmentation $\Theta^\Gamma_0 \to \mathbb{Z}$ is the composition of $\hat{\varepsilon}$ with the augmentation $\Theta^{\Gamma_\sigma}_0 \to \mathbb{Z}$.

Then the statement follows from the comparison theorem [20, 2.2.6] of fundamental homological algebra. In fact, the properties imply that $\hat{\varepsilon}$ is a chain map of resolutions lifting the identity on $\mathbb{Z}$. An inverse is given by the canonical inclusion $\Theta^{\Gamma_\sigma}_\bullet \to \Theta^\Gamma_\bullet$, and since the composition is unique up to chain homotopy equivalence, it must be homotopic to the identity. □

The group $\Gamma_\sigma$ acts diagonally on $\Theta^\Gamma_1 \cong \mathbb{Z}\Gamma \otimes_\mathbb{Z} \mathbb{Z}\Gamma$, and trivially on $\mathbb{Z}$, so we can consider $\Theta^\Gamma_1 \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$.

**Corollary 12.** *Denote by $\varepsilon$ is the augmentation from $\mathbb{Z}\Gamma$ to $\mathbb{Z}$. Let a cycle $(\sum_i (a_i \otimes_\mathbb{Z} b_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1)$ in $\Theta^\Gamma_1 \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$ be given, where $a_i, b_i \in \Gamma$. The ensuing element in $\mathrm{H}_1(\Gamma_\sigma;\mathbb{Z})$ is then given by*

$\sum_i \varepsilon(a_i)\, \overline{\varepsilon_\ell(a_i)^{-1}\varepsilon_\ell(b_i)}$.

This expression makes sense because $\varepsilon_\ell(a_i)$ is invertible in $\mathbb{Z}\Gamma_\sigma$.
Note that the cycle condition on $\sum\limits_i (a_i \otimes_\mathbb{Z} b_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1$ says that $\sum\limits_i (b_i - a_i) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 = 0$, which means that $\sum_i a_i$ is equivalent to $\sum_i b_i$ modulo $\mathbb{Z}\Gamma_\sigma$.

*Proof.* Using the lemma 10, we just need to apply the map

$$(\varepsilon_\ell \otimes_\mathbb{Z} \varepsilon_\ell) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 : (\mathbb{Z}\Gamma \otimes_\mathbb{Z} \mathbb{Z}\Gamma) \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z} \to (\mathbb{Z}\Gamma_\sigma \otimes_\mathbb{Z} \mathbb{Z}\Gamma_\sigma) \otimes_{\mathbb{Z}\Gamma_\sigma} \mathbb{Z}$$

to get

$$\begin{aligned} \sum (\varepsilon_\ell \otimes_\mathbb{Z} \varepsilon_\ell \otimes_{\mathbb{Z}\Gamma_\sigma} 1)(a_i \otimes_\mathbb{Z} b_i \otimes_{\mathbb{Z}\Gamma_\sigma} 1) &= \sum (\varepsilon_\ell(a_i) \otimes_\mathbb{Z} \varepsilon_\ell(b_i)) \otimes_{\mathbb{Z}\Gamma_\sigma} 1 \\ &= \sum (1 \otimes_\mathbb{Z} \varepsilon_\ell(a_i)^{-1}\varepsilon_\ell(b_i)) \otimes_{\mathbb{Z}\Gamma_\sigma} \varepsilon(a_i). \end{aligned}$$

In bar notation, this is $\sum \left[\varepsilon_\ell(a_i)^{-1}\varepsilon_\ell(b_i)\right] \otimes_{\mathbb{Z}\Gamma_\sigma} \varepsilon(a_i)$, and is mapped to

$$\sum_i \varepsilon(a_i)\, \overline{\varepsilon_\ell(a_i)^{-1}\varepsilon_\ell(b_i)} \in \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}),$$

and to $\sum_i \varepsilon(a_i)\, \varepsilon_\ell(a_i)^{-1}\varepsilon_\ell(b_i) \mod [\Gamma_\sigma, \Gamma_\sigma]$ by the isomorphism into the abelianization of $\Gamma_\sigma$ described in [5, page 36]. □

2.0.2. *The mass formula for the Euler characteristic.*
We will use the Euler characteristic to check the geometry of the quotient $\Gamma\backslash X$. Recall the following definitions and proposition, which we include for the reader's convenience.

**Definition 13** (Euler characteristic)**.** *Suppose $\Gamma'$ is a torsion-free group. Then we define its Euler characteristic as*

$$\chi(\Gamma') = \sum_i (-1)^i \dim \mathrm{H}_i(\Gamma'; \mathbb{Q}).$$

*Suppose further that $\Gamma'$ is a torsion-free subgroup of finite index in a group $\Gamma$. Then we define the* Euler characteristic *of $\Gamma$ as*

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma : \Gamma']}.$$

*This is well-defined because of* [5, IX.6.3]*.*

**Definition 14** (Equivariant Euler characteristic)**.** *Suppose $X$ is a $\Gamma$-complex such that*

(1) *every isotropy group $\Gamma_\sigma$ is of finite homological type;*
(2) *$X$ has only finitely many cells mod $\Gamma$.*

*Then we define the* $\Gamma$-equivariant Euler characteristic *of $X$ as*

$$\chi_\Gamma(X) := \sum_\sigma (-1)^{\dim \sigma} \chi(\Gamma_\sigma),$$

*where $\sigma$ runs over the orbit representatives of cells of $X$.*

**Proposition 15** ([5, IX.7.3 e'])**.** *Suppose $X$ is a $\Gamma$-complex such that $\chi_\Gamma(X)$ is defined. If $\Gamma$ is virtually torsion-free, then $\Gamma$ is of finite homological type and $\chi(\Gamma) = \chi_\Gamma(X)$.*

Let now $\Gamma$ be $\mathrm{PSL}_2\big(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}\big)$. Then the above proposition applies to $X$ taken to be Flöge's (or still, Mendoza's) $\Gamma$-equivariant deformation retract of $\mathcal{H}$. Using $\chi(\Gamma_\sigma) = \frac{1}{\mathrm{card}(\Gamma_\sigma)}$ for $\Gamma_\sigma$ finite, the fact that the singular points have stabilizer $\mathbb{Z}^2$, and the torsion-free Euler characteristic

$$\chi(\mathbb{Z}^2) = \sum_i (-1)^i \mathrm{rank}_\mathbb{Z}(\mathrm{H}_i\, \mathbb{Z}^2) = 1 - 2 + 1 = 0,$$

we get the formula

$$\chi(\Gamma) = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

where $\sigma$ runs over the orbit representatives of cells of $X$ with finite stabilizers.

**Proposition 16.** *The Euler characteristic* $\chi(\Gamma)$ *vanishes.*

***Remark* 17.** This, together with the formula

$$0 = \chi(\Gamma) = \chi_\Gamma(X) = \sum_{\sigma} (-1)^{\dim \sigma} \frac{1}{\mathrm{card}(\Gamma_\sigma)},$$

allows to check the joint data of the geometry of the fundamental domain, cell stabilizers and cell identifications.

*Proof of proposition 16.* Denote by $\zeta_K$ the Dedekind $\zeta$-function associated to the number field $K := \mathbb{Q}\left[\sqrt{-m}\right]$. Brown [5, below (IX.8.7)] deduces the following from Harder's result [10, p. 453]:

$$\chi(SL_n(\mathcal{O}_K)) = \prod_{j=2}^{n} \zeta_K(1-j),$$

so especially we have $\chi(SL_2(\mathcal{O}_K)) = \zeta_K(-1)$. As any cell $\sigma$ in the interior of hyperbolic space has a stabilizer $SL_2(\mathcal{O}_K)_\sigma$ of twice the cardinality of $\Gamma_\sigma$, it follows that

$$\chi(\Gamma) = \frac{1}{2}\chi(SL_2(\mathcal{O}_K)) = \frac{1}{2}\zeta_K(-1).$$

Using the functional equation of $\zeta_K$ [13] and the fact that $K$ has no real places because it is imaginary quadratic, we get $\zeta_K(-1) = 0$. $\square$

***Remark* 18.** One can prove the above proposition without using the Dedekind zeta function. This alternative proof applies to *any* cofinite arithmetically defined subgroup $\Gamma$ of $\mathrm{PSL}(2, \mathbb{C})$. It is the main theorem of Harder's article on the Gauss-Bonnet theorem [10] that the Euler characteristic is the covolume of $\Gamma$ with respect to the Euler-Poincaré form $\mu$ on $\mathcal{H}$, i. e. $\chi(\Gamma) = \int_Y d\mu$, where $Y$ is a fundamental domain for the action of $\Gamma$ on $\mathcal{H}$. This extends the classical Gauss-Bonnet theorem from the theory of the Euler-Poincaré form, see [16, paragraph 3] (here the theorem is hidden as the existence assertion of the Euler-Poincaré measure) to non-cocompact but cofinite discrete subgroups. The measure $\mu$ is a fundamental datum associated to the symmetric space, without reference to any discrete group. In [16, paragraph 3,2a] it is shown that $\mu = 0$ on any odd-dimensional space. Since $\dim \mathcal{H} = 3$, we have $\chi(\Gamma) = 0$.

## 3. Computations of the integral homology of $\mathrm{PSL}_2\left(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}\right)$

Throughout this section, we assume the action on the homology coefficients to be trivial, which is realized by our cell structure. We mean $\mathbb{Z}$-coefficients wherever we do not mention the coefficients. We will always label the singular point in the fundamental domain by $s$; and we use the notation

$$\otimes_\sigma := \otimes_{\mathbb{Z}[\Gamma_\sigma]}.$$

We will write $\mathcal{D}_2$ for the Klein four group, $\mathcal{S}_3$ for the permutation group on three objects and $\mathcal{A}_4$ for the alternating group on four objects. We have $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{\mathbb{Q}[\sqrt{-m}]}) = \mathrm{PSL}_2(\mathbb{Z}[\omega])$ with $\omega := \sqrt{-m}$ in the cases $m = 5, 6, 10, 13$. The coordinates in Hyperbolic space of the vertices of the fundamental domains have been computed by Bianchi [4]. There, they are listed up to complex conjugation for $m = 5, 6, 15$; and for $m = 10, 13$, the reader has to divide out the reflection called *riflessione impropria* by Bianchi.

### 3.1. $m = 13$.

We make the following definitions.

$$A := \pm\begin{pmatrix} 9 & 7\omega \\ \omega & -10 \end{pmatrix}, \qquad B := \pm\begin{pmatrix} -2-\omega & 2-\omega \\ 4 & 2+1\omega \end{pmatrix},$$

$$C := \pm\begin{pmatrix} -1-\omega & 8-\omega \\ 3 & 1+2\omega \end{pmatrix}, \qquad D := \pm\begin{pmatrix} 5 & 2\omega \\ \omega & -5 \end{pmatrix},$$

$$E := \pm\begin{pmatrix} -\omega & 6 \\ 2 & \omega \end{pmatrix}, \qquad J := \pm\begin{pmatrix} & 1 \\ -1 & \end{pmatrix},$$

$$S := \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, \qquad K := \pm\begin{pmatrix} 11+4\omega & -17+7\omega \\ -8+\omega & -10-3\omega \end{pmatrix},$$

$$M := \pm\begin{pmatrix} 4-2\omega & 12+\omega \\ 4+\omega & -4+2\omega \end{pmatrix}, \qquad U := \pm\begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix},$$

$$V := \pm\begin{pmatrix} -\omega & 6-\omega \\ 2 & 2+\omega \end{pmatrix}, \qquad W := \pm\begin{pmatrix} 14-\omega & 13+6\omega \\ 2\omega & -12+\omega \end{pmatrix},$$

$$P := V^{-1}D, \qquad T := P^{-1}S^2,$$

$$R := TU^{-1}S^2U.$$

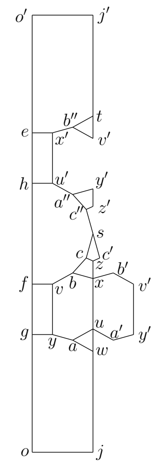



**Figure 1:** The fundamental domain for $m = 13$

We observe the relations $T = CKCA(CKC)^{-1}$, $V^{-1} = CAC^{-1}M$ and $S^2 = BS^{-1}BS$. The matrix $U$ acts as a vertical translation by $-\omega$ on this fundamental domain. There are seventeen orbits of vertices, which have the following stabilizers.

$$\begin{array}{rcl}
\Gamma_o & = & \langle J \mid J^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_a & = & \langle S^{-1}BS \mid (S^{-1}BS)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_b & = & \Gamma_c = \langle M \mid M^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_u & = & \langle B \mid B^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_v & = & \langle D \mid D^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_f & = & \langle D, E \mid D^2 = E^2 = (DE)^2 = 1\rangle \cong D_2, \\
\Gamma_h & = & \langle E, AU^{-1}JU \mid E^2 = (AU^{-1}JU)^2 = (EAU^{-1}JU)^2 = 1\rangle \cong D_2, \\
\Gamma_e & = & \langle A, U^{-1}JU \mid A^3 = (U^{-1}JU)^2 = (AU^{-1}JU)^2 = 1\rangle \cong S_3, \\
\Gamma_g & = & \langle J, T \mid J^2 = T^3 = (JT)^2 = 1\rangle \cong S_3, \\
\Gamma_t & = & \langle R, U^{-1}SU \mid R^2 = (U^{-1}SU)^3 = (RU^{-1}SU)^2 = 1\rangle \cong S_3, \\
\Gamma_w & = & \langle B, S \mid B^2 = S^3 = (BS)^2 = 1\rangle \cong S_3, \\
\Gamma_j & = & \langle S \mid S^3 = 1\rangle \cong \mathbb{Z}/3, \\
\Gamma_x & = & \Gamma_z = \langle CAC^{-1} \mid (CAC^{-1})^3 = 1\rangle \cong \mathbb{Z}/3, \\
\Gamma_y & = & \langle T \mid T^3 = 1\rangle \cong \mathbb{Z}/3, \\
\Gamma_s & = & \langle V, W \mid VW = WV\rangle \cong \mathbb{Z}^2.
\end{array}$$

There are twenty-eight orbits of edges. The edge stabilizers of isomorphy type $\mathbb{Z}/2$ are given on the chosen edge orbit representatives as

$$\begin{array}{rcl}
\Gamma_{(f,v)} & = & \langle D \mid D^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(h,u')} & = & \langle EAU^{-1}JU \mid (EAU^{-1}JU)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(t,b'')} & = & \langle R \mid R^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(w,a)} & = & \langle S^{-1}BS \mid (S^{-1}BS)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(b,c)} & = & \langle M \mid M^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(a'',c'')} & = & \langle C^{-1}S^{-1}BSC \mid (C^{-1}S^{-1}BSC)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(v',t)} & = & \langle RU^{-1}SU \mid (RU^{-1}SU)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(w,u)} & = & \langle B \mid B^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(h,e)} & = & \langle AU^{-1}JU \mid (AU^{-1}JU)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(g,f)} & = & \langle DE \mid (DE)^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(f,h)} & = & \langle E \mid E^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(o,g)} & = & \langle J \mid J^2 = 1\rangle \cong \mathbb{Z}/2, \\
\Gamma_{(o',e)} & = & \langle U^{-1}JU \mid (U^{-1}JU)^2 = 1\rangle \cong \mathbb{Z}/2.
\end{array}$$

and the edge stabilizers of isomorphy type $\mathbb{Z}/3$ are given on the chosen edge orbit representatives as

$$\begin{array}{lcl}
\Gamma_{(e,x')} & = & \langle A | A^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(x,z)} & = & \langle CAC^{-1} | (CAC^{-1})^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(g,y)} & = & \langle T | T^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(j,w)} & = & \langle S | S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(t,j')} & = & \langle U^{-1}SU | (U^{-1}SU)^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(y',z')} & = & \langle KCA(KC)^{-1} | (KCA(KC)^{-1})^3 = 1 \rangle \cong \mathbb{Z}/3.
\end{array}$$

We use the identifications $C \cdot x' = x$, $U \cdot j' = j$, $C \cdot y' = y$ and $K \cdot z = z'$. We find nine edge orbits with the trivial stabilizer, thirteen edge orbit representatives with stabilizer type $\mathbb{Z}/2$, and six with stabilizer type $\mathbb{Z}/3$. The singular vertex has stabilizer type $\mathbb{Z}^2$, and there are six vertex orbit representatives with stabilizer type $\mathbb{Z}/2$, two with $\mathcal{D}_2$, four with $\mathcal{S}_3$ and four with $\mathbb{Z}/3$. Furthermore, there are twelve orbits of faces with trivial stabilizers.
The above data gives the $\Gamma$-equivariant Euler characteristic of $X$, in accordance with remark 17:

$$\chi_\Gamma(X) = 6 \cdot \frac{1}{2} + 4 \cdot \frac{1}{3} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{6} - 9 - 13 \cdot \frac{1}{2} - 6 \cdot \frac{1}{3} + 12 = 0.$$

3.1.1. *$E^1$-page for $m = 13$.*
We obtain for the row $q = 0$ in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^{17} \xleftarrow{d^1_{1,0}} \mathbb{Z}^{28} \xleftarrow{d^1_{2,0}} \mathbb{Z}^{12},$$

where the only occurring elementary divisor is 1, with multiplicity sixteen for $d^1_{1,0}$, and with multiplicity ten for $d^1_{2,0}$.

3.1.2. *Odd rows of the $E^1$-term.*
For odd $q$, the morphism

$$\bigoplus_{\sigma \in \Gamma \backslash X^0} \mathrm{H}_q(\Gamma_\sigma) \xleftarrow{d^1_{1,q}} \bigoplus_{\sigma \in \Gamma \backslash X^1} \mathrm{H}_q(\Gamma_\sigma)$$

is on the 2-primary part a homomorphism

$$(\mathbb{Z}/2)^{q+13} \longleftarrow (\mathbb{Z}/2)^{13}$$

given by the $q$+13-by-13 matrix

$$(d^1_{1,q})_{(2)} = \left( \begin{array}{ccccccccccccc}
1 & & & & & & & & & 1 & & & \\
 & & & & & & & & & \vdots & & & \\
 & & & & & & & & & 1 & 1 & & \\
 & 1 & & & & & & & 1 & & & & \\
 & \vdots & & & & & & & & & & & \\
 & 1 & & & & & & & & & 1 & & \\
 & & & & & & & & & & & 1 & 1 \\
 & & & & & & & & 1 & & & & 1 \\
 & & & & & & & & & 1 & & 1 & \\
 & & 1 & & & & 1 & & & & & & \\
 & 1 & & & & & & 1 & & & & & \\
1 & & & & & & 1 & & & & & & \\
 & & & 1 & & & & 1 & & & & & \\
 & & & 1 & & 1 & & & & & & & \\
 & & 1 & & 1 & & & & & & & & \\
 & & & & 1 & 1 & & & & & & &
\end{array} \right),$$

where we replace each occurence of the dots " $\vdots$ " by $\frac{q-1}{2}$ lines with a "1" in the column of the dots and zeroes elsewhere. This is due to the diagonal map

$$(\mathbb{Z}/2)^{\frac{q+3}{2}} \cong \mathrm{H}_q(\mathcal{D}_2) \leftarrow \mathrm{H}_q(\mathbb{Z}/2) \cong \mathbb{Z}/2, \qquad \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \leftarrow 1,$$

which is induced by the inclusion $\mathcal{D}_2 \leftarrow \mathbb{Z}/2$ hitting the product of the two fixed generators of $\mathcal{D}_2$. Therefore, we have to distinguish the case $q = 1$, where $d^1_{1,q}$ has rank 12, and the case $q \geqslant 3$, where it has rank 13.

On the 3-primary part, $d^1_{1,q}$ is a homomorphism

$$\begin{cases} (\mathbb{Z}/3)^4 \longleftarrow (\mathbb{Z}/3)^6 & \text{for } q \equiv 1 \mod 4, \\ (\mathbb{Z}/3)^8 \longleftarrow (\mathbb{Z}/3)^6 & \text{for } q \equiv 3 \mod 4. \end{cases}$$

It is given by the matrix

$$(d^1_{1,q})_{(3)} = \begin{array}{c|cccccc} & (e,x') & (g,y) & (x,z) & (y',z') & (j,w) & (t,j') \\ \hline e & -\alpha & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & -1 & 0 & 0 & 0 \\ g & 0 & -\alpha & 0 & 0 & 0 & 0 \\ y & 0 & 1 & 0 & -1 & 0 & 0 \\ z & 0 & 0 & 1 & 1 & 0 & 0 \\ j & 0 & 0 & 0 & 0 & -1 & 1 \\ w & 0 & 0 & 0 & 0 & \alpha & 0 \\ t & 0 & 0 & 0 & 0 & 0 & -\alpha, \end{array}$$

where $\alpha = 1$ for $q \equiv 3 \mod 4$ and $\alpha = 0$ else. This matrix has full rank 6 (injectivity) for $q \equiv 3 \mod 4$, and rank 4 (surjectivity) for $q \equiv 1 \mod 4$. For $q = 1$, there is an additional module $\mathrm{H}_1(\Gamma_s) \cong \mathbb{Z}^2$ on the target side, which can not be hit because the edge stabilizers are only torsion.

***Remark* 19.** So, the 3-torsion in $\mathrm{H}_1(\Gamma)$ has already been killed by the $d^1$ differential. This shows that there is no injection

$$\mathrm{H}_1(\mathrm{PSL}_2(\mathbb{Z})) \to \mathrm{H}_1(\Gamma).$$

We verify this fact by considering the generator of the 3-torsion in $\mathrm{H}_1(\mathrm{PSL}_2(\mathbb{Z}))$, which is induced by the matrix $S$. In the group $\Gamma$ for $m = 13$, the matrix $S$ of order three is subject to the relation $S^2 = BS^{-1}BS$ where $B$ is a matrix of order two defined above. The right hand side of this equation can be simplified to the unit element when we pass it to $\Gamma^{ab} \cong \mathrm{H}_1(\Gamma)$. So, the above non-injectivity is based on the fact that $S$ does not survive abelianizing $\Gamma$ whilst it survives abelianizing $\mathrm{PSL}_2(\mathbb{Z})$.

3.1.3. *The rows with $q$ even.*
There is a zero map arriving at $\bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}_q(\Gamma_\sigma) \cong (\mathbb{Z}/2)^q$ for $q$ bigger than 2, and respectively at

$$\bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}_2(\Gamma_\sigma) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

3.1.4. *The $E^2$-page for $m = 13$.* In the rows with $q \geqslant 2$, $E^2_{p,q}$ is concentrated in the columns $p = 0$ and $p = 1$ given as follows:

$$\begin{array}{ll|cc} q = 4k+1, & q \geqslant 5 & (\mathbb{Z}/2)^q & (\mathbb{Z}/3)^2 \\ q \text{ even}, & q \geqslant 4 & (\mathbb{Z}/2)^q & 0 \\ q = 4k+3, & q \geqslant 3 & (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^q & 0 \\ \dots & & \dots & \dots \\ q = 2 & & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

In the rows $q = 0$ and $q = 1$, $E^2_{p,q}$ is concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd} q = 1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 & (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 & 0 \\ q = 0 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 \arrow[ull, "d^2"'] \end{tikzcd}$$

3.1.5. *Computation of the differential $d^2$.*
The only nontrivial $d^2$-arrow is determined on the $E^0$-level by the following maps connecting $E^0_{2,0}$ with $E^0_{0,1}$:

$$\begin{tikzcd}
\bigoplus\limits_{\sigma\in\Gamma\backslash X^0} \Theta_1 \otimes_\sigma \mathbb{Z} & \bigoplus\limits_{\sigma\in\Gamma\backslash X^1} \Theta_1 \otimes_\sigma \mathbb{Z} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta\otimes 1"] & \\
& \bigoplus\limits_{\sigma\in\Gamma\backslash X^1} \Theta_0 \otimes_\sigma \mathbb{Z} & \bigoplus\limits_{\sigma\in\Gamma\backslash X^2} \Theta_0 \otimes_\sigma \mathbb{Z} \arrow[l, "1\otimes\delta"']
\end{tikzcd}$$

where $d_\Theta$ is the differential of the bar resolution $\Theta_\bullet$ for $\Gamma$, and $\delta$ is the differential of Flöge's cellular complex. The generators of the abelian group $E^2_{2,0} \cong \mathbb{Z}^2$ are represented by the face $(c, s, c', z)$ and the union of two faces $(b, x, b', v', y', a', u, a, y, v) =: F$, whose quotients by $\Gamma$ are homeomorphic to 2-spheres. We observe the edge identifications $CAC^{-1} \cdot (c, z) = (c', z)$, $V^{-1} \cdot (s, c) = (s, c')$, $CAC^{-1} \cdot (b, x) = (b', x)$, $V^{-1} \cdot (b, v) = (b', v')$, $P \cdot (y, v) = (y', v')$, $S^2 \cdot (a, y) = (a', y')$, and $B \cdot (a, u) = (a', u)$.
The above $d^2$-arrow is induced by

$$\delta\left((c, s, c', z)\right) = (CAC^{-1} - 1) \cdot (c, z) + (V^{-1} - 1) \cdot (s, c)$$

and

$$\delta\left((b, x, b', v', y', a', u, a, y, v)\right) = (CAC^{-1} - 1) \cdot (x, b) + (V^{-1} - 1) \cdot (b, v) + (P - 1) \cdot (v, y) + (S^2 - 1) \cdot (y, a) + (B - 1) \cdot (a, u).$$

The lift $1 \otimes_F 1$ in $E^0_{2,0}$ of the generator of $E^2_{2,0}$ represented by
$F = (b, x, b', v', y', a', u, a, y, v)$ is mapped as follows:

$$\begin{tikzcd}
\begin{array}{c} (1, CAC^{-1}) \otimes_b 1 - (1, CAC^{-1}) \otimes_x 1 \\ +(1, V^{-1}) \otimes_v 1 - (1, V^{-1}) \otimes_b 1 \\ +(1, P) \otimes_y 1 - (1, P) \otimes_v 1 \\ +(1, S^2) \otimes_a 1 - (1, S^2) \otimes_y 1 \\ +(1, B) \otimes_u 1 - (1, B) \otimes_a 1 \end{array} & \begin{array}{c} (1, CAC^{-1}) \otimes_{(x,b)} 1 \\ +(1, V^{-1}) \otimes_{(b,v)} 1 \\ +(1, P) \otimes_{(v,y)} 1 \\ +(1, S^2) \otimes_{(y,a)} 1 \\ +(1, B) \otimes_{(a,u)} 1 \end{array} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta\otimes 1"] & \\
& \begin{array}{c} (CAC^{-1} - 1) \otimes_{(x,b)} 1 \\ +(V^{-1} - 1) \otimes_{(b,v)} 1 \\ +(P - 1) \otimes_{(v,y)} 1 \\ +(S^2 - 1) \otimes_{(y,a)} 1 \\ +(B - 1) \otimes_{(a,u)} 1 \end{array} & 1 \otimes_F 1 \arrow[l, "1\otimes\delta"']
\end{tikzcd}$$

*The passage to $E^1$.* We attribute the symbols $t_\sigma$ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$:

$$\begin{aligned}
t_x &:= -(1, CAC^{-1}) \otimes_x 1, \\
t_b &:= (1, CAC^{-1}) \otimes_b 1 - (1, V^{-1}) \otimes_b 1, \\
t_v &:= (1, V^{-1}) \otimes_v 1 - (1, P) \otimes_v 1, \\
t_y &:= (1, P) \otimes_y 1 - (1, S^2) \otimes_y 1, \\
t_a &:= (1, S^2) \otimes_a 1 - (1, B) \otimes_a 1, \\
t_u &:= (1, B) \otimes_u .
\end{aligned}$$

With the formula in our corollary 12, we find the classes $\bar{t_\sigma}$ in $\mathrm{H}_1(\Theta_* \otimes_\sigma \mathbb{Z})$ as follows:
As $V^{-1}M = CAC^{-1}$ and $\Gamma_b = \langle M | \quad M^2 = 1\rangle$,

$$t_b = [CAC^{-1}] \otimes_b 1 - [V^{-1}] \otimes_b 1 = [V^{-1}M] \otimes_b 1 - [V^{-1}] \otimes_b 1$$

gives the cycle

$$\overline{VV^{-1}M} - \overline{VV^{-1}} = \overline{M} \in \langle \overline{M} | \quad 2\overline{M} = 0\rangle \cong \mathrm{H}_1(\Gamma_b; \mathbb{Z}).$$

As $V^{-1} = PD$ and $\Gamma_v = \langle D | \quad D^2 = 1 \rangle$,

$$t_v = [V^{-1}] \otimes_v 1 - [P] \otimes_v 1 = [PD] \otimes_v 1 - [P] \otimes_v 1$$

gives the cycle

$$\overline{P^{-1}PD} - \overline{P^{-1}P} = \overline{D} \in \langle \overline{D} | \quad 2\overline{D} = 0 \rangle \cong \mathrm{H}_1(\Gamma_v; \mathbb{Z}).$$

As $S^2 = BS^{-1}BS$ and $\Gamma_a = \langle S^{-1}BS | \quad (S^{-1}BS)^2 = 1 \rangle$,

$$t_a = [S^2] \otimes_a 1 - [B] \otimes_a 1 = [BS^{-1}BS] \otimes_a 1 - [B] \otimes_a 1$$

gives the cycle

$$\overline{B^{-1}BS^{-1}BS} - \overline{B^{-1}B} = \overline{S^{-1}BS} \in \langle \overline{S^{-1}BS} | \quad 2\overline{S^{-1}BS} = 0 \rangle \cong \mathrm{H}_1(\Gamma_a; \mathbb{Z}).$$

Finally, $t_u = [B] \otimes_u 1$ gives the cycle

$$\overline{B} \in \langle \overline{B} | \quad 2\overline{B} = 0 \rangle \cong \mathrm{H}_1(\Gamma_u; \mathbb{Z}).$$

The term $E^2_{0,1}$ having no 3-torsion, the 3-torsion part $\bar{t_x} + \bar{t_y}$ of the above sum makes no contribution to the image of $d^2$.

The 2-torsion part, $\overline{t_b} + \overline{t_a} + \overline{t_v} + \overline{t_u}$, equals the image

$$d^1_{1,1}(\overline{t_{(b,c)}} + \overline{t_{(c'',a'')}} + \overline{t_{(v,f)}} + \overline{t_{(f,h)}} + \overline{t_{(h,u')}}),$$

where $\overline{t_\sigma}$ stands for the generator of $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}) \cong \mathbb{Z}/2$. Hence it makes no contribution neither, and we obtain $d^2(F) = 0$.

The lift $1 \otimes_{(c,s,c',z)} 1$ of the generator $(c, s, c', z)$ is mapped as follows:

$$\begin{tikzcd}
\begin{array}{c} (1, CAC^{-1}) \otimes_z 1 \\ -(1, CAC^{-1}) \otimes_c 1 \\ +(1, V^{-1}) \otimes_c 1 \\ -(1, V^{-1}) \otimes_s 1 \end{array} & \begin{array}{c} (1, CAC^{-1}) \otimes_{(c,z)} 1 \\ +(1, V^{-1}) \otimes_{(s,c)} 1 \end{array} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta \otimes 1"] & \\
& \begin{array}{c} (CAC^{-1} - 1) \otimes_{(c,z)} 1 \\ +(V^{-1} - 1) \otimes_{(s,c)} 1 \end{array} & 1 \otimes_{(c,s,c',z)} 1 \arrow[l, "1\otimes\delta"']
\end{tikzcd}$$

*The passage to $E^1$.* We attribute the symbols $t_\sigma$ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$:

$$\begin{aligned} t_z &:= (1, CAC^{-1}) \otimes_z 1, \\ t_c &:= (1, V^{-1}) \otimes_c 1 - (1, CAC^{-1}) \otimes_c 1, \\ t_s &:= -(1, V^{-1}) \otimes_s 1. \end{aligned}$$

With the formula in our corollary 12, we find the classes $\bar{t_\sigma}$ in $\mathrm{H}_1(\Theta_* \otimes_\sigma \mathbb{Z})$ as follows:

$$t_z = [CAC^{-1}] \otimes_z 1$$

gives the cycle

$$\overline{CAC^{-1}} \in \langle \overline{CAC^{-1}} | \quad 3\overline{CAC^{-1}} = 0 \rangle \cong \mathrm{H}_1(\Gamma_z; \mathbb{Z}).$$

As $V^{-1}M = CAC^{-1}$ and $\Gamma_c = \langle M | \quad M^2 = 1 \rangle$,

$$t_c = [V^{-1}] \otimes_c 1 - [CAC^{-1}] \otimes_c 1 = [V^{-1}] \otimes_c 1 - [V^{-1}M] \otimes_c 1$$

gives the cycle

$$\overline{VV^{-1}} - \overline{VV^{-1}M} = -\overline{M} \in \langle \overline{M} | \quad 2\overline{M} = 0 \rangle \cong \mathrm{H}_1(\Gamma_c; \mathbb{Z}).$$

Finally,

$$t_s = -[V^{-1}] \otimes_s 1$$

gives the cycle

$$\overline{V} \in \langle \overline{V}, \overline{W} \rangle \cong \mathrm{H}_1(\Gamma_s; \mathbb{Z}) \cong \mathbb{Z}^2.$$

The term $E^2_{0,1}$ having no 3-torsion, the 3-torsion part $\overline{t_z}$ of the above sum makes no contribution to the image of $d^2$.
However the 2-torsion part, $\overline{t_c} = \overline{M}$, passes to the $E^2$-page because no chain of edges can have the single point $c$ as its boundary. Furthermore, $\overline{V}$ is one of the generators of the free part of $E^2_{0,1} \cong \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2$, so we obtain $d^2((c, s, c', z)) = \overline{M} + \overline{V}$, which is of infinite order and has the following property: There is no element $\eta \in E^2_{0,1}$ with $k\eta = \overline{M} + \overline{V}$ for an integer $k > 1$. As we have seen that $d^2(F) = 0$, we obtain the quotient

$$E^3_{0,1} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

Hence we obtain for integral homology the following short exact sequences:

$$\begin{cases} 0 \to (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to (\mathbb{Z}/3)^2 \to 0, & q = 4k+2, \\ 0 \to (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 4k+1, \\ 0 \to (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 4k+4, \\ 0 \to (\mathbb{Z}/3)^2 \oplus (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 4k+3, \\ 0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_2(\Gamma; \mathbb{Z}) \to \mathbb{Z} \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2 \to 0, & \\ 0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_1(\Gamma; \mathbb{Z}) \to \mathbb{Z}^2 \to 0. & \end{cases}$$

We will resolve the ambiguity of the extension $\mathrm{H}_2(\Gamma; \mathbb{Z})$ by a reflection like the one on [15, page 587], for which we have to recompute the spectral sequence with $\mathbb{Z}/2$-coefficients.

3.1.6. *The spectral sequence for $m = 13$ with $\mathbb{Z}/2$-coefficients.*
From [15, lemma 4.2(3)] we compute by elementary means that $\mathrm{H}_q(\mathcal{S}_3; \mathbb{Z}/2) \cong \mathbb{Z}/2$, $\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2) \cong \mathbb{Z}/2$ and $\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{q+1}$ for all $q \geqslant 0$. Also by elementary means, we get

$$\mathrm{H}_q(\mathbb{Z}^2; \mathbb{Z}/2) \cong \begin{cases} 0, & q \geqslant 3, \\ \mathbb{Z}/2, & q = 2, \\ (\mathbb{Z}/2)^2, & q = 1. \end{cases}$$

3.1.7. *The $E^1$-page with $\mathbb{Z}/2$-coefficients.*
We can apply the functor $- \otimes \mathbb{Z}/2$ to the row $q = 0$ and obtain in the columns $p = 0, 1, 2$:

$$(\mathbb{Z}/2)^{17} \xleftarrow{d^1_{1,0}} (\mathbb{Z}/2)^{28} \xleftarrow{d^1_{2,0}} (\mathbb{Z}/2)^{12}.$$

The rest of this row are zeroes. The matrix $d^1_{1,0}$ has rank 16 and the matrix $d^1_{2,0}$ has rank 10.
In the rows with $q > 0$, the differential $d^1$ is given by a single arrow $d^1_{1,q}$ from $E^1_{1,q} = (\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2))^{13} \oplus (\mathrm{H}_q(\mathbb{Z}/3; \mathbb{Z}/2))^6 \cong (\mathbb{Z}/2)^{13}$ to

$$E^1_{0,q} = \mathrm{H}_q(\mathbb{Z}^2; \mathbb{Z}/2) \oplus (\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2))^6 \oplus (\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2))^2 \oplus (\mathrm{H}_q(\mathcal{S}_3; \mathbb{Z}/2))^4,$$

and the rest of these rows are zeroes. For $q = 1$, we have $d^1_{1,1}$ of rank 12 arriving at $E^1_{0,1} \cong (\mathbb{Z}/2)^{16}$. For $q \geqslant 3$, we have $d^1_{1,q}$ of rank 13 arriving at $E^1_{0,q} \cong (\mathbb{Z}/2)^{12+2q}$. For $q = 2$, we have $d^1_{1,2}$ of rank 13 arriving at $E^1_{0,2} \cong (\mathbb{Z}/2)^{17}$. The only difficulty in seeing this is to compute the maps from $\mathrm{H}_q(\mathbb{Z}/2; \mathbb{Z}/2)$ to $\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}/2)$ induced by the inclusions $f : \mathbb{Z}/2 \to \mathcal{D}_2$. For this task we take the resolutions of $\mathbb{Z}/2 \cong \langle t \,|\, t^2 = 1 \rangle$ and $\mathcal{D}_2 \cong \langle D, E \,|\, D^2 = E^2 = (DE)^2 = 1 \rangle$ proposed by [15] and compute the chain map induced by extending $f$ to a ring homomorphism $f : \mathbb{Z}[\mathbb{Z}/2] \to \mathbb{Z}[\mathcal{D}_2]$. We can then apply the functor $- \otimes_{\mathbb{Z}[G]} \mathbb{Z}/2$ to this chain map (where $\mathbb{Z}/2$ is the trivial $\mathbb{Z}[G]$-module for $G = \mathbb{Z}/2, \mathcal{D}_2$ ).

3.1.8. *The $E^2$-page for $m = 13$ with $\mathbb{Z}/2$-coefficients.* We obtain in the rows with $q \geqslant 2$ the $E^2$-term concentrated in the column $p = 0$,

$$\begin{array}{c|c} q \geqslant 3 & (\mathbb{Z}/2)^{2q-1} \\ q = 2 & (\mathbb{Z}/2)^4, \end{array}$$

and in the rows $q = 0$, $q = 1$ it is concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd} q=1 & (\mathbb{Z}/2)^4 & \mathbb{Z}/2 & 0 \\ q=0 & \mathbb{Z}/2 & (\mathbb{Z}/2)^2 & (\mathbb{Z}/2)^2. \arrow[ull, "d^2_{2,0}"'] \end{tikzcd}$$

*Computation of the differential $d^2_{2,0}$.* The basis $\{(c, s, c', z),\ F\}$ of $E^2_{2,0}$ with $\mathbb{Z}$-coefficients induces a basis of $E^2_{2,0}$ with $\mathbb{Z}/2$-coefficients. The Universal Coefficient Theorem yields an isomorphism from $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Z}/2$ to $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2)$, which we will use to transfer the elements $\overline{t_\sigma} \in \mathrm{H}_1(\Gamma_\sigma; \mathbb{Z})$ computed in subsection 3.1.5 to $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2)$.
For $d^2_{2,0}((c, s, c', z))$ the computation is as follows. As $\overline{t_c}$ generates $\mathrm{H}_1(\Gamma_c; \mathbb{Z}) \cong \mathbb{Z}/2$, it is transferred to the generator of $\mathrm{H}_1(\Gamma_c; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Since $\overline{t_s}$ can be completed with a second element to a $\mathbb{Z}$-basis of $\mathrm{H}_1(\Gamma_s; \mathbb{Z}) \cong \mathbb{Z}^2$, it is transferred to a nontrivial element of $\mathrm{H}_1(\Gamma_s; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^2$. The element $\overline{t_z}$ vanishes because $\mathrm{H}_1(\Gamma_z; \mathbb{Z}) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/3 \otimes \mathbb{Z}/2 = 0$. The sum $\overline{t_c} + \overline{t_s}$ is quotiented to a nontrivial element on the $E^2$-page because $\mathrm{H}_1(\Gamma_s; \mathbb{Z}/2)$ is not hit by the $d^1$-differential. So $d^2_{2,0}(\langle (c, s, c', z) \rangle) \cong \mathbb{Z}/2$.
For $d^2_{2,0}(F)$, the computation is as follows. The 3-torsion vanishing when tensoring with $\mathbb{Z}/2$, the 3-torsion part $\bar{t_x} + \bar{t_y}$ of the sum makes no contribution to the image of $d^2$. The 2-torsion part, $\overline{t_b} + \overline{t_a} + \overline{t_v} + \overline{t_u}$, equals the image

$$d^1_{1,1}(\overline{t_{(b,c)}} + \overline{t_{(c'',a'')}} + \overline{t_{(v,f)}} + \overline{t_{(f,h)}} + \overline{t_{(h,u')}}),$$

where $\overline{t_\sigma}$, $\sigma \in \{b, a, v, u, (b, c), (c'', a''), (v, f), (f, h), (h, u')\}$ is the generator of $\mathrm{H}_1(\Gamma_\sigma; \mathbb{Z}/2) \cong \mathbb{Z}/2$. Hence it makes no contribution neither, and we obtain $d^2(F) = 0$. Thus $d^2_{2,0}$ has rank 1.

Then, the $E^3 = E^\infty$-page yields immediately

$$\mathrm{H}_q(\Gamma; \mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^{2q-1}, & q \geqslant 3, \\ (\mathbb{Z}/2)^6, & q = 2, \\ (\mathbb{Z}/2)^5, & q = 1, \end{cases}$$

and we use the Universal Coefficient Theorem in the form

$$\mathrm{H}_2(\Gamma; \mathbb{Z}/2) \cong \mathrm{H}_2(\Gamma; \mathbb{Z}) \otimes (\mathbb{Z}/2) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(\mathrm{H}_1(\Gamma; \mathbb{Z}), \mathbb{Z}/2)$$

to conclude, after an analogous computation with $\mathbb{Z}/3$-coefficients,

$$\mathrm{H}_q(\Gamma; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1, \\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus (\mathbb{Z}/3)^2 \oplus \mathbb{Z}/2, & q = 2, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k+3, \quad k \geqslant 0, \\ (\mathbb{Z}/2)^q, & q = 4k+4, \quad k \geqslant 0, \\ (\mathbb{Z}/2)^q, & q = 4k+1, \quad k \geqslant 1, \\ (\mathbb{Z}/2)^q \oplus (\mathbb{Z}/3)^2, & q = 4k+2, \quad k \geqslant 1. \end{cases}$$

### 3.2. $m = 5$.

We will make use of the following matrices:

$$A := \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad B := \pm\begin{pmatrix} -\omega & 2 \\ 2 & \omega \end{pmatrix},$$
$$M := \pm\begin{pmatrix} -\omega & 4 \\ 1 & \omega \end{pmatrix}, \qquad S := \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix},$$
$$U := \pm\begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix}, \qquad V := \pm\begin{pmatrix} -\omega & 2-\omega \\ 2 & 2+\omega \end{pmatrix},$$
$$W := \pm\begin{pmatrix} 6-\omega & 5+2\omega \\ 2\omega & \omega-4 \end{pmatrix},$$

which are subject to the relations $UMU^{-1} = A$, $UWS(UW)^{-1} = S$, $WABW^{-1} = MB$ and $S = ABV$.

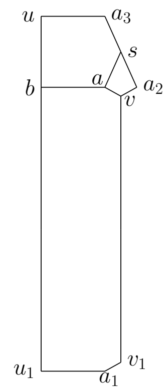



**Figure 2:** The fundamental domain for $m = 5$

There are five orbits of vertices, with stabilizers

$$\begin{aligned}
\Gamma_b &= \langle A, B \mid A^2 = B^2 = 1\rangle \cong \mathcal{D}_2,\\
\Gamma_u &= \langle B, M \mid B^2 = M^2 = 1\rangle \cong \mathcal{D}_2,\\
\Gamma_a &= \langle AB \mid AB^2 = 1\rangle \cong \mathbb{Z}/2,\\
\Gamma_v &= \langle S \mid S^3 = 1\rangle \cong \mathbb{Z}/3,\\
\Gamma_s &= \langle V, W \mid VW = WV\rangle \cong \mathbb{Z}^2
\end{aligned}$$

and identifications $UW \cdot a = a_1$, $V^{-1} \cdot a = a_2$, $S^2 \cdot a = a_2$, $U \cdot u = u_1$ and $UW \cdot v = v_1$. There are seven orbits of edges, with stabilizers

$$\begin{aligned}
\Gamma_{(b,a)} &= \langle AB \mid AB^2 = 1\rangle \cong \mathbb{Z}/2,\\
\Gamma_{(v,v_1)} &= \langle S \mid S^3 = 1\rangle \cong \mathbb{Z}/3,\\
\Gamma_{(a_3,u)} &= \langle MB \mid MB^2 = 1\rangle \cong \mathbb{Z}/2,\\
\Gamma_{(u,b)} &= \langle B \mid B^2 = 1\rangle \cong \mathbb{Z}/2,\\
\Gamma_{(u_1,b)} &= \langle A \mid A^2 = 1\rangle \cong \mathbb{Z}/2;
\end{aligned}$$

$(a, v)$ and $(a, s)$ having the trivial stabilizer. There are three orbits of faces, with trivial stabilizers. The above data gives the $\Gamma$-equivariant Euler characteristic of $X$:

$$\chi_\Gamma(X) = \frac{1}{2} + \frac{1}{3} + 2\cdot\frac{1}{4} - 2 - 4\cdot\frac{1}{2} - \frac{1}{3} + 3 = 0,$$

in accordance with remark 17.

3.2.1. *The zeroth row of the $E^1$-page.* This row identifies with the cellular chain complex of the quotient complex $\Gamma\backslash X$.
We obtain for the row $q = 0$ in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d^1_{1,0}} \mathbb{Z}^7 \xleftarrow{d^1_{2,0}} \mathbb{Z}^3$$

where 1 is the only elementary divisor of the differential matrices, with multiplicity four for $d^1_{1,0}$, and multiplicity two for $d^1_{2,0}$. The homology of $\Gamma\backslash X$ is generated in degree 1 by the loop represented by the edge $(v, v_1)$, and in degree 2 by the quotient of the face $(a_2, s, a, v)$, which is homeomorphic to a 2-sphere.

3.2.2. *Odd rows of the $E^1$-page.* We start by investigating the morphism

$$\mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^5 \xleftarrow{d^1_{1,1}} \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^4$$

and the morphism

$$\mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{q+4} \xleftarrow{d^1_{1,q}} \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^4$$

for $q \geqslant 3$. On the 3-torsion, $d^1_{1,q}$ is zero.
On the 2-torsion, $d^1_{1,q}$ it is given by the matrix

$$(d^1_{1,q})_{(2)} = \begin{array}{c|cccc} & (b,a) & (a_3,u) & (u,b) & (u_1,b) \\ \hline a & 1 & -1 & 0 & 0 \\ b & -1 & 0 & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b & -1 & 0 & 1 & 0 \\ u & 0 & 1 & -1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ u & 0 & 1 & 0 & -1, \end{array}$$

where we fill in $\frac{q-1}{2}$ zero rows into each dotted line, except that in the columns with a "-1" both above and below the dots, we write "-1" into all entries of this column which are between the two "-1"'s. Thus $d^1_{1,1}$ has rank 3 and $d^1_{1,q}$ has rank 4 for $q \geqslant 3$.

3.2.3. *Even rows of the $E^1$-term.* There is a zero map arriving at $E^1_{0,2} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$.
For $q \geqslant 4$, there is a zero map arriving at $E^1_{0,q} \cong (\mathbb{Z}/2)^q$.
The rest of the $E^1$-page are zeroes.

3.2.4. *The $E^2$-page for $m = 5$.*
In the rows with $q \geqslant 2$, the $E^2$-page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l|cc} q \geqslant 4 \text{ even} & (\mathbb{Z}/2)^q & 0 \\ q \geqslant 3 \text{ odd} & (\mathbb{Z}/2)^q \oplus \mathbb{Z}/3 & \mathbb{Z}/3 \\ q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd} q = 1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 \\ q = 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \arrow[ull, "d^2"'] \end{tikzcd}$$

Let us compute the only nontrivial $d^2$-arrow. The generator of $E^2_{2,0}$ comes from the 2-cell $(a_2, s, a, v)$. Among its vertices, we have the identifications $S^2 \cdot a = a_2$ and $V^{-1} \cdot a = a_2$, where the matrices $V$ of infinite order stabilizes the singular point $s$, and the matrix $S$ of order three stabilizes the point $v$. The lift $1 \otimes_{(a_2,s,a,v)} 1$ of the generator of $E^2_{2,0}$ is mapped as follows in the $E^0$-page:

$$\begin{tikzcd} \begin{array}{c} (V^{-1},1) \otimes_s 1 - (V^{-1},1) \otimes_a 1 \\ +(1,S^2) \otimes_v 1 - (1,S^2) \otimes_a 1 \end{array} & \begin{array}{c} (V^{-1},1) \otimes_{(a,s)} 1 \\ +(1,S^2) \otimes_{(a,v)} 1 \end{array} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta \otimes 1"] & \\ & \begin{array}{c} 1 \otimes_{(a,s)} 1 - V^{-1} \otimes_{(a,s)} 1 \\ +S^2 \otimes_{(a,v)} 1 - 1 \otimes_{(a,v)} 1 \end{array} & 1 \otimes_{(a_2,s,a,v)} 1 \arrow[l, "1\otimes\delta"'] \end{tikzcd}$$

It passes to

$$(\overline{V}, 2\overline{S}, \overline{AB}) \in \langle \overline{V}, \overline{W} \rangle \oplus \langle \overline{S} \,|\, 3\overline{S} = 0 \rangle \oplus (\mathbb{Z}/2)^2 \cong E^2_{0,1},$$

which is of infinite order and has the following property: There is no element $\eta \in E^2_{0,1}$ with $k\eta = (\overline{V}, 2\overline{S}, \overline{AB})$ for an integer $k > 1$. So,

$$E^3_{0,1} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3.$$

Thus the $E^\infty$-page yields the following short exact sequences:

$$\begin{cases} 0 \to (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma;\mathbb{Z}) \to \mathbb{Z}/3 \to 0 & q \geqslant 4 \text{ even}, \\ 0 \to \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma;\mathbb{Z}) \to 0 & q \geqslant 3 \text{ odd}, \\ 0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_2(\Gamma;\mathbb{Z}) \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to 0, & \\ 0 \to \mathbb{Z} \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_1(\Gamma;\mathbb{Z}) \to \mathbb{Z} \to 0. & \end{cases}$$

To resolve the ambiguity of the extension $\mathrm{H}_2(\Gamma;\mathbb{Z})$, we compute $\mathrm{H}_q(\Gamma;\mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^4 & q = 1, \\ (\mathbb{Z}/2)^5 & q = 2, \\ (\mathbb{Z}/2)^{2q-1} & q \geqslant 3 \end{cases}$ and get the result

$$\mathrm{H}_q(\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 & q = 1, \\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 & q = 2, \\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q & q \geqslant 3. \end{cases}$$

***Remark* 20.** There are the following checks of the computations we made. Let us look at the low term short exact sequence

$$0 \longrightarrow E^\infty_{0,1} \longrightarrow \Gamma^{\mathrm{ab}} \xrightarrow{\rho} E^\infty_{1,0} \longrightarrow 0$$

of the spectral sequence. We have $E^\infty_{1,0} = \mathrm{H}_1(\Gamma\backslash X) = (\pi_1(\Gamma\backslash X))^{\mathrm{ab}}$, and the projection $\rho$ is the abelianization of the map $\Gamma \to \pi_1(\Gamma\backslash X)$ given as follows. Choose a fixed base point $x \in X$. For every $\gamma \in \Gamma$, choose a continuous path in $X$ from $x$ to $\gamma x$. This gives a well-defined loop in $\Gamma\backslash X$ since $X$ is contractible. As Flöge shows, an inspection of the complex $X$ and the associated stabilizer groups and identifications yields, together with [1, theorem 4.5], a presentation of $\Gamma$ by means of generators and relations. In order to get $\Gamma^{\mathrm{ab}}$, we use the presentation computed by Flöge for $m = 5, 6, 10$, and the presentation computed by Swan [18] for $m = 15$. Then, we compute the group $E^\infty_{0,1} = E^3_{0,1}$ as the kernel of the projection $\rho$.
For $m = 5$, this check looks as follows.
The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{A}, \overline{B}, \overline{S}, \overline{U}, \overline{V} : 2\overline{A} = 0, 2\overline{B} = 0, 3\overline{S} = 0 \rangle$. The fundamental group of the quotient space being free, only the parabolic elements $U$ and $V$ can define nontrivial loops in the quotient space. The element $U$ generates a nontrivial loop, whilst $V$ generates a trivial loop.
So it follows that $E^\infty_{0,1} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3$, generated by $\overline{V}, \overline{A}, \overline{B}$ and $\overline{S}$. This is consistent with the computation above, involving the detailed analysis of the $d^2$-differential.

### 3.3. $m = 10$.

We will use the following definitions:

$$A := \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad B := \pm\begin{pmatrix} -\omega & 3 \\ 3 & \omega \end{pmatrix},$$

$$C := \pm\begin{pmatrix} -1-\omega & 4-\omega \\ 2 & 1+\omega \end{pmatrix}, \qquad D := \pm\begin{pmatrix} \omega - 1 & -4 \\ 3 & 1+\omega \end{pmatrix},$$

$$L := \pm\begin{pmatrix} \omega & 3 \\ 3 & -\omega \end{pmatrix}, \qquad R := \pm\begin{pmatrix} 5+\omega & 2\omega - 3 \\ \omega - 3 & -4-\omega \end{pmatrix},$$

$$S := \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix}, \qquad U := \pm\begin{pmatrix} 1 & \omega \\ & 1 \end{pmatrix},$$

$$V := \pm\begin{pmatrix} 1-\omega & 5 \\ 2 & 1+\omega \end{pmatrix}, \qquad W := \pm\begin{pmatrix} 11 & 5\omega \\ 2\omega & -9 \end{pmatrix},$$

$$Y := \pm\begin{pmatrix} \omega - 2 & -5 \\ 3 & 2+\omega \end{pmatrix}.$$

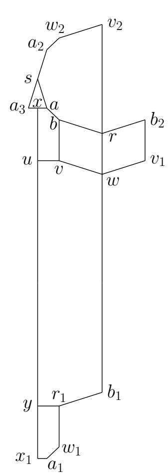



**Figure 3:** The fundamental domain for $m = 10$

The matrix $U$ acts as a vertical translation by $-\omega$ on this fundamental domain. There are nine orbits of vertices, labelled $a, b, r, u, v, w, x, y, s$. We have the following identifications: $UWa = a_1$, $Wa = a_2$, $Va = a_3$; $S^{-1}v = v_1$, $U^{-1}Dv = v_2$; $Dw = w_1$, $U^{-1}Dw = w_2$; $Db = b_1$, $Cb = b_2$; $Dr = r_1$; $UWx = x_1$. The stabilizers of the vertex orbit representatives are

$$\begin{array}{lcl}
\Gamma_a & = & \Gamma_b = \langle R \mid R^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_w & = & \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_y & = & \langle A, L \mid A^2 = L^2 = (AL)^2 = 1 \rangle \cong \mathcal{D}_2, \\
\Gamma_u & = & \langle A, B \mid A^2 = B^2 = (AB)^2 = 1 \rangle \cong \mathcal{D}_2, \\
\Gamma_r & = & \langle C \mid C^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_v & = & \langle AB \mid (AB)^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_x & = & \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_s & = & \langle V, W \mid VW = WV \rangle \cong \mathbb{Z}^2.
\end{array}$$

There are fifteen orbits of edges, labelled $(b, v), (r, w), (b, r), (v, w), (a_2, w_2)$, $(y, r_1), (x, a), (u, y), (a, b), (u, v), (a, s), (w, b_1), (r, v_2), (y, x_1), (x, u)$.
Amongst their stabilizers only

$$\begin{array}{lcl}
\Gamma_{(a_2,w_2)} & = & \Gamma_{a_2} = W^{-1}\Gamma_a W = \langle W^{-1}RW \mid (W^{-1}RW)^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(a,b)} & = & \Gamma_a = \Gamma_b = \langle R \mid R^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(w,b_1)} & = & \Gamma_{b_1} = \Gamma_w = \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(y,r_1)} & = & \Gamma_{r_1} = D\Gamma_r D^{-1} = \langle AL = DCD^{-1} \mid (DCD^{-1})^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(u,v)} & = & \Gamma_v = \langle AB \mid (AB)^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(r,v_2)} & = & \Gamma_{v_2} = \Gamma_r = \langle C \mid C^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(y,x_1)} & = & \Gamma_{x_1} = UW\Gamma_x(UW)^{-1} = \langle L \mid L^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(x,u)} & = & \Gamma_x = \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(u,y)} & = & \langle A \mid A^2 = 1 \rangle \cong \mathbb{Z}/2
\end{array}$$

are nontrivial. Furthermore, there are seven orbits of faces, with trivial stabilizers.
With the above information on the isomorphy types of the cell stabilizers, we get the $\Gamma$-equivariant Euler characteristic of $X$:

$$\chi_\Gamma(X) = 3 \cdot \frac{1}{3} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{2} - 3 \cdot \frac{1}{3} - 6 \cdot \frac{1}{2} - 6 + 7 = 0,$$

in accordance with remark 17.

3.3.1. *The row $q = 0$ in the $E^1$-page for $m = 10$.*
We obtain for the row $q = 0$ in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^9 \xleftarrow{d^1_{1,0}} \mathbb{Z}^{15} \xleftarrow{d^1_{2,0}} \mathbb{Z}^7,$$

where 1 is the only elementary divisor of the differential matrices, with multiplicity eight for $d^1_{1,0}$, and multiplicity five for $d^1_{2,0}$. The rest of this row are zeroes.

3.3.2. *Odd rows of the $E^1$-term.*
For odd $q$, the morphism

$$\bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}_q(\Gamma_\sigma) \xleftarrow{d^1_{1,q}} \bigoplus_{\sigma \in \Gamma\backslash X^1} \mathrm{H}_q(\Gamma_\sigma)$$

is for $q \geqslant 3$ of the form

$$(\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^{q+6} \longleftarrow (\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^6.$$

For $q = 1$, we have to add $\mathrm{H}_1(\Gamma_s) \cong \mathbb{Z}^2$ on the target side of the morphism $d^1_{1,q}$, but the incoming torsion must reach it trivially.
On the 3-primary part, $d^1_{1,q}$ is given by the matrix

$$(d^1_{1,q})_{(3)} = \begin{array}{c|ccc}
 & (a,b) & (Db,w) & (Wa, U^{-1}Dw) \\ \hline
a & -1 & 0 & -1 \\
w & 0 & 1 & 1 \\
b & 1 & -1 & 0.
\end{array}$$

This matrix has rank 2, so its image is isomorphic to $(\mathbb{Z}/3)^2$ and its kernel is of type $\mathbb{Z}/3$.
On the 2-primary part, $d^1_{1,q}$ is for odd $q$ given by the matrix

$$(d^1_{1,q})_{(2)} = \begin{array}{c|cccccc} & (y,r_1) & (u,v) & (r,v_2) & (y,x_1) & (x,u) & (u,y) \\ \hline u & 0 & -1 & 0 & 0 & 0 & -1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u & 0 & -1 & 0 & 0 & 1 & 0 \\ y & -1 & 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ y & -1 & 0 & 0 & -1 & 0 & 0 \\ x & 0 & 0 & 0 & 1 & -1 & 0 \\ r & 1 & 0 & -1 & 0 & 0 & 0 \\ v & 0 & 1 & 1 & 0 & 0 & 0, \end{array}$$

where, as in the computation for $m = 13$, we fill in $\frac{q-1}{2}$ zero rows into each of the two dotted lines, except that in the columns with a "-1" both above and below the dots, we write "-1" into all entries of this column which are between the two "-1"'s. The above matrix $(d^1_{1,q})_{(2)}$ has rank 5 for $q = 1$, and full rank 6 for $q \geqslant 3$.

3.3.3. *The rows with $q$ even.* These rows are given by zero maps into $\bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}_q(\Gamma_\sigma) \cong (\mathbb{Z}/2)^q$ for $q > 2$, respectively into $\bigoplus_{\sigma \in \Gamma\backslash X^0} \mathrm{H}_2(\Gamma_\sigma) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$.

3.3.4. *The $E^2$-page for $m = 10$.*
In the rows with $q \geqslant 2$, the $E^2$-page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l|cc} q \geqslant 4 \text{ even} & (\mathbb{Z}/2)^q & 0 \\ q \geqslant 3 \text{ odd} & (\mathbb{Z}/2)^q \oplus \mathbb{Z}/3 & \mathbb{Z}/3 \\ q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0 \end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd} q=1 & \mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 \\ q=0 & \mathbb{Z} & \mathbb{Z}^2 & \mathbb{Z}^2 \arrow[ull, "d^2"'] \end{tikzcd}$$

3.3.5. *Computation of the differential $d^2$.*
The generators of the abelian group $E^2_{2,0} \cong \mathbb{Z}^2$ are represented by the 2-cell $(a, s, a_3, x)$ and the union of two 2-cells $(v_1, b_2, r, b, v, w)$, whose quotients by $\Gamma$ are homeomorphic to 2-spheres. On the vertices of $(a, s, a_3, x)$ , we have the identifications $B \cdot a = a_3$ and $V \cdot a = a_3$, where the matrix $B$ fixes $x$ and the matrix $V$ fixes $s$. For $(v_1, b_2, r, b, v, w)$, we have the identifications of vertices $Cb = b_2$, $Cr = r$, $S^2v = v_1$ and $S^2w = w$; and we pay particular attention to the matrix $CR = S^2AB$ identifying the edge $(b, v) \cong (b_2, v_1)$. Thus the only nontrivial $d^2$-arrow is induced by

$$\delta((a, s, a_3, x)) = (a, s) + V \cdot (s, a) + B \cdot (a, x) + (x, a)$$

and

$$\delta((v_1, b_2, r, b, v, w)) = (b, r) - C \cdot (b, r) + CR \cdot (b, v) + S^2 \cdot (v, w) - (v, w) - (b, v).$$

The lift $1 \otimes_{(v_1,b_2,r,b,v,w)} 1$ of the generator obtained from $(v_1, b_2, r, b, v, w)$ is mapped as follows:

$$\begin{tikzcd}
\begin{array}{c}(C,1)\otimes_r 1 - (C,1)\otimes_b 1\\ +(1,CR)\otimes_v 1 - (1,CR)\otimes_b 1\\ +(1,S^2)\otimes_w 1 - (1,S^2)\otimes_v 1\end{array} & \begin{array}{c}(C,1)\otimes_{(b,r)} 1\\ +(1,CR)\otimes_{(b,v)} 1\\ +(1,S^2)\otimes_{(v,w)} 1\end{array} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta\otimes 1"] & \\
 & \begin{array}{c}1\otimes_{(b,r)} 1 - C\otimes_{(b,r)} 1\\ +CR\otimes_{(b,v)} 1 - 1\otimes_{(b,v)} 1\\ +S^2\otimes_{(v,w)} 1 - 1\otimes_{(v,w)} 1\end{array} & 1\otimes_{(v_1,b_2,r,b,v,w)} 1 \arrow[l, "1\otimes\delta"']
\end{tikzcd}$$

We obtain $d^2_{2,0}(\langle(v_1, b_2, r, b, v, w)\rangle) \cong \mathbb{Z}/3$.

The lift $1 \otimes_{(a,s,a_3,x)} 1$ of the generator obtained from $(a, s, a_3, x)$ is mapped

$$\begin{tikzcd}
\begin{array}{c}(V,1)\otimes_s 1 - (V,1)\otimes_a 1\\ +(1,B)\otimes_x 1 - (1,B)\otimes_a 1\end{array} & \begin{array}{c}(V,1)\otimes_{(a,s)} 1\\ +(1,B)\otimes_{(a,x)} 1\end{array} \arrow[l, "1\otimes\delta"'] \arrow[d, "d_\Theta\otimes 1"] & \\
 & \begin{array}{c}1\otimes_{(a,s)} 1 - V\otimes_{(a,s)} 1\\ +B\otimes_{(a,x)} 1 - 1\otimes_{(a,x)} 1\end{array} & 1\otimes_{(a,s,a_3,x)} 1 \arrow[l, "1\otimes\delta"']
\end{tikzcd}$$

We attribute the symbols $t_\sigma$ to the part of this sum lying in $\Theta_1 \otimes_\sigma \mathbb{Z}$,

$$\begin{array}{rcl} t_s &:=& (V,1)\otimes_s 1,\\ t_x &:=& (1,B)\otimes_x 1,\\ t_a &:=& -(V,1)\otimes_a 1 - (1,B)\otimes_a 1. \end{array}$$

We find the class $\overline{t_s} = -\overline{V} \in \langle \overline{V}, \overline{W}\rangle = \Gamma_s^{\mathrm{ab}} \cong \mathrm{H}_1(\Gamma_s;\mathbb{Z}) \cong \mathbb{Z}^2$, which is a generator of the free part of $E^1_{0,1}$. It can not be the image of a torsion element from $E^1_{1,1} = (\mathbb{Z}/3)^3 \oplus (\mathbb{Z}/2)^2$. Therefore, it is preserved when passing from $E^1_{0,1}$ to $E^2_{0,1}$. The cycles $\overline{t_x}$ and $\overline{t_a}$ are torsion, so the fact that $\overline{t_s}$ is a generator of the free part determines that the image $d^2_{2,0}(\langle(a, s, a_3, x)\rangle)$ is of infinite order and has the following property: There is no element $\eta \in E^2_{0,1} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2$ with $k\eta = d^2_{2,0}(\langle(a, s, a_3, x)\rangle)$ for an integer $k > 1$. Together with the isomorphy $d^2_{2,0}(\langle(v_1, b_2, r, b, v, w)\rangle) \cong \mathbb{Z}/3$, we obtain

$$E^3_{0,1} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2.$$

Thus the $E^\infty$-page gives the following short exact sequences:

$$\begin{cases} 0 \to (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma;\mathbb{Z}) \to \mathbb{Z}/3 \to 0, & \text{for } q \geqslant 4 \text{ even},\\ 0 \to \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q \to \mathrm{H}_q(\Gamma;\mathbb{Z}) \to 0, & \text{for } q \geqslant 3 \text{ odd},\\ 0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_2(\Gamma;\mathbb{Z}) \to \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to 0, & \\ 0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_1(\Gamma;\mathbb{Z}) \to \mathbb{Z}^2 \to 0. & \end{cases}$$

From here, we easily see the results,

$$\mathrm{H}_q(\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^3 \oplus (\mathbb{Z}/2)^2, & q = 1,\\ \mathbb{Z}^2 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 2,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^q, & q \geqslant 3; \end{cases}$$

except for the ambiguity in the 3-torsion and the 2-torsion of the short exact sequence for $\mathrm{H}_2(\Gamma;\mathbb{Z})$. To resolve it, we compute $\mathrm{H}_q(\Gamma;\mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^{2q-1}, & q \geqslant 3\\ (\mathbb{Z}/2)^6, & q = 2,\\ (\mathbb{Z}/2)^5, & q = 1, \end{cases}$ and also homology with $\mathbb{Z}/3$-coefficients.

***Remark* 21.** For $m = 10$, the check introduced in remark 20 takes the following form. The abelianization is the group

$$\Gamma^{\text{ab}} \cong \langle \overline{A}, \overline{B}, \overline{D}, \overline{U}, \overline{W} : 2\overline{A} = 2\overline{B} = 0 \rangle.$$

The elements of infinite order are $D$, $U$ and $W$. The elements $U$ and $U^{-1}D$ give the cycles generating $\mathrm{H}_1(\Gamma\backslash X)$, whilst $W$ generates a trivial loop. So it follows that $E^\infty_{0,1} = \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, generated by $\overline{W}, \overline{A}$ and $\overline{B}$. This is consistent with the computation above.

3.4. $m = 6$.

The matrix $U := \pm\left(\begin{smallmatrix} 1 & \omega \\ & 1 \end{smallmatrix}\right)$ performs a vertical translation by $-\omega$ of the fundamental domain for $\Gamma$. The following matrices occur in the cell stabilizers.

$$A := \pm\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}, \qquad B := \pm\begin{pmatrix} -1-\omega & 2-\omega \\ 2 & 1+\omega \end{pmatrix},$$
$$R := \pm\begin{pmatrix} -\omega & 5-\omega \\ 1 & 1+\omega \end{pmatrix}, \qquad S := \pm\begin{pmatrix} & -1 \\ 1 & 1 \end{pmatrix},$$
$$V := \pm\begin{pmatrix} 1-\omega & 3 \\ 3 & 1+\omega \end{pmatrix}, \qquad W := \pm\begin{pmatrix} 7 & 3\omega \\ 2\omega & -5 \end{pmatrix}.$$

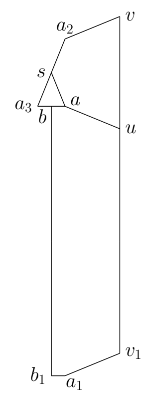



**Figure 4:** The fundamental domain for $m = 6$

There are five orbits of vertices, labelled $b, a, u, v, s$, with stabilizers

$$\begin{array}{rcl}
\Gamma_u &=& \langle B, S \mid B^2 = S^3 = (BS)^3 = 1 \rangle \cong \mathcal{A}_4, \\
\Gamma_v &=& \langle B, \quad R \mid B^2 = R^3 = (BR)^3 = 1 \rangle \cong \mathcal{A}_4, \\
\Gamma_{v_1} &=& \langle UBU^{-1}, S \mid (UBU^{-1})^2 = S^3 = (UBU^{-1}S)^3 = 1 \rangle \cong \mathcal{A}_4, \\
\Gamma_a &=& \langle SB \mid (SB)^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{a_2} &=& \langle RB \mid (RB)^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_b &=& \langle A \mid A^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_s &=& \langle V, W \mid VW = WV \rangle \cong \mathbb{Z}^2,
\end{array}$$

and identifications $UW \cdot a = a_1$, $W \cdot a = a_2$, $V \cdot a = a_3$, $A \cdot a = a_3$, $UW \cdot b = b_1$ and $U \cdot v = v_1$. There are seven orbits of edges, labelled $(b, a)$, $(a, s)$, $(a, u)$, $(u, v)$, $(a_2, v)$, $(b, b_1)$ and $(u, v_1)$, amongst whose stabilizers only

$$\begin{array}{rcl}
\Gamma_{(a_2,v)} &=& \langle RB \mid (RB)^3 = 1 \rangle = \Gamma_{a_2} \cong \mathbb{Z}/3, \\
\Gamma_{(u,v_1)} &=& \langle S \mid S^3 = 1 \rangle \cong \mathbb{Z}/3, \\
\Gamma_{(a,u)} &=& \langle SB \mid (SB)^3 = 1 \rangle = \Gamma_a \cong \mathbb{Z}/3, \\
\Gamma_{(u,v)} &=& \langle B \mid B^2 = 1 \rangle \cong \mathbb{Z}/2, \\
\Gamma_{(b,b_1)} &=& \langle A \mid A^2 = 1 \rangle = \Gamma_b = \Gamma_{b_1} \cong \mathbb{Z}/2
\end{array}$$

are nontrivial; and three orbits of faces with trivial stabilizers. The above data gives the $\Gamma$-equivariant Euler characteristic of $X$:

$$\chi_\Gamma(X) = \frac{1}{12} + \frac{1}{12} + \frac{1}{3} + \frac{1}{2} - 1 - 1 - \frac{1}{3} - \frac{1}{3} - \frac{1}{3} - \frac{1}{2} - \frac{1}{2} + 3 = 0,$$

in accordance with remark 17.

3.4.1. *Zeroth row of the* $E^1$*-term.*
We obtain in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d^1_{1,0}} \mathbb{Z}^7 \xleftarrow{d^1_{2,0}} \mathbb{Z}^3$$

where 1 is the only occurring elementary divisor of the differential matrices, with multiplicity four for $d^1_{1,0}$, and multiplicity two for $d^1_{2,0}$. The homology of this sequence is generated by the cycle $(b, b_1)$ in degree one and by the face $(a, s, a_3, b)$ in degree two.

3.4.2. *Even rows of the $E^1$-term.*
The even rows are the zero map to $E^1_{0,2} \cong \mathbb{Z} \oplus (\mathbb{Z}/2)^2$, and to $E^1_{0,q} \cong (\mathrm{H}_q(\mathcal{A}_4))^2$ for the degree $q \geqslant 4$.

3.4.3. *Odd rows of the $E^1$-term.*
The map $d^1_{1,q}$ is on the 2-primary part induced by the inclusion of $\Gamma_{(u,v)} \cong \mathbb{Z}/2$ into $\Gamma_v$ and $\Gamma_u$ which are of isomorphy type $\mathcal{A}_4$. By [15, lemma 4.5(2)], every inclusion of $\mathbb{Z}/2$ into $\mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on $\mathrm{H}_1$. So the morphism

$$\mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus (\mathbb{Z}/3)^3 \xleftarrow{d^1_{1,1}} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$$

has $\mathbb{Z}/2$-rank 0 on the 2-primary part, and

$$\mathbb{Z}/3 \oplus \mathbb{Z}/2 \oplus (\mathrm{H}_q(\mathcal{A}_4))^2 \xleftarrow{d^1_{1,q}} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^3$$

in the odd rows of degree $q \geqslant 3$ has $\mathbb{Z}/2$-rank 1 on the 2-primary part.

On the 3-primary part, $d^1_{1,q}$ is for all odd $q$ given by the following rank 2 matrix.

$$(d^1_{1,q})_{(3)} = \begin{array}{c|ccc} & (a,u) & (a_2,v) & (u,v_1) \\ \hline a & -1 & -1 & 0 \\ u & 1 & 0 & -1 \\ v & 0 & 1 & 1. \end{array}$$

In order to determine it, we make use of the following facts.
First, by [15, lemma 4.5], each of the occurring group inclusions induces an injection in homology. So we have to determine the relative positions of the images coming from the edges in each direct summand over the points. In order to find out if cancelling occurs between terms with positive and negative signs, let us look at the following diagram. The symbol $\Delta W$ denotes the isomorphism given by conjugation with $W$, $\delta$ denotes an inner automorphism, $\iota$ denotes any canonical inclusion, and the arrows emanating from $\mathbb{Z}/3$ are labeled with the image of the canonical generator.

$$\begin{tikzcd}
 & & \Gamma_{(a_2,v)} \arrow[dll, "\mathrm{id}"'] \arrow[drr, "\iota"] & & \\
\Gamma_{a_2} \arrow[d, "\Delta W"'] & & \mathbb{Z}/3 \arrow[ll, "RB"] \arrow[rr, "RB"'] \arrow[u, "RB"'] \arrow[dll, "SB"] \arrow[ddll, "SB"] \arrow[ddl, "SB"] \arrow[dd, "S"] \arrow[ddr, "S"] \arrow[ddrr, "S"] \arrow[drr, "SUBU^{-1}"] & & \Gamma_v \arrow[d, "\Delta U"] \\
\Gamma_a & & & & \Gamma_{v_1} \\
\Gamma_{(a,u)} \arrow[u, "\mathrm{id}"] \arrow[r, "\iota"'] & \Gamma_u \arrow[r, "\delta"'] & \Gamma_u & \Gamma_{(u,v_1)} \arrow[l, "\iota"'] \arrow[r, "\iota"'] & \Gamma_{v_1} \arrow[u, "\delta"']
\end{tikzcd}$$

Applying homology $\mathrm{H}_q$ for odd $q$ and taking into account that the fact that inner automorphisms act trivially on homology, we get a similar slightly smaller commutative diagram. One can then unambiguously identify all occurring groups $\mathrm{H}_q(\mathbb{Z}/3) \cong \mathbb{Z}/3$ and its images in $\mathrm{H}_q(\mathcal{A}_4)$ with the "abstract" $\mathrm{H}_q(\mathbb{Z}/3) \cong \mathbb{Z}/3$ in the middle. This gives a basis for the 3-primary parts of the source and a subspace of the image. In this basis, the 3-primary map is given by the following matrix, followed by an injection which does not influence the homology.

3.4.4. *The $E^2$-page for $m = 6$.*
In the rows with $q \geqslant 2$, the $E^2$-page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{l|cc}
q = 6k+2, q \geqslant 8 & (\mathbb{Z}/2)^{2k+2} & 0 \\
q = 6k+1, q \geqslant 7 & (\mathbb{Z}/2)^{2k} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\
q = 6k+6 & (\mathbb{Z}/2)^{2k} & 0 \\
q = 6k+5 & (\mathbb{Z}/2)^{2k+4} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\
q = 6k+4 & (\mathbb{Z}/2)^{2k} & 0 \\
q = 6k+3 & (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\
q = 2 & \mathbb{Z} \oplus (\mathbb{Z}/2)^2 & 0
\end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd}
q = 1 & \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 & (\mathbb{Z}/2)^2 \oplus \mathbb{Z}/3 & 0 \\
q = 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \arrow[ull]
\end{tikzcd}$$

3.4.5. *The $E^3 = E^\infty$-term.* For the calculation of the $d^2$-differential, we have

$$\begin{aligned}
\delta(a, s, a_3, b) &= (a_3, s) + (s, a) + (a, b) + (b, a_3) \\
&= (V \cdot a, s) + (s, a) + (a, b) + (b, A \cdot a) \\
&= V \cdot (a, s) - (a, s) - (b, a) + A \cdot (b, a),
\end{aligned}$$

$$\begin{aligned}
(1 \otimes \delta)(1 \otimes_{(a,s,a_3,b)} 1) &= 1 \otimes_{V\cdot(a,s)} 1 - 1 \otimes_{(a,s)} 1 - 1 \otimes_{(b,a)} 1 + 1 \otimes_{A\cdot(b,a)} 1 \\
&= (V - 1) \otimes_{(a,s)} 1 + (A - 1) \otimes_{(b,a)} 1 \\
&= (d_\Theta \otimes 1) \left((1, V) \otimes_{(a,s)} 1 + (1, A) \otimes_{(b,a)} 1\right) \\
&= (d_\Theta \otimes 1) \left([V] \otimes_{(a,s)} 1 + [A] \otimes_{(b,a)} 1\right).
\end{aligned}$$

We then get

$$(1 \otimes \delta) \left([V] \otimes_{(a,s)} 1 + [A] \otimes_{(b,a)} 1\right) = [V] \otimes_s 1 - [V] \otimes_a 1 + [A] \otimes_a 1 - [A] \otimes_b 1.$$

As $[V] \otimes_s 1$ and $[W] \otimes_s 1$ represent the generators of the torsion-free part of $E^2_{0,1} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$, we see that the above computed element of $E^0_{0,1}$ represents an element $\nu \in E^2_{0,1}$ of infinite order with the following property: There is no element $\eta \in E^2_{0,1}$ with $k\eta = \nu$ for an integer $k > 1$. So, $E^3_{0,1} \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$ and $E^3_{2,0} = 0$.

3.4.6. *The short exact sequences.*
We thus obtain for integral homology the following short exact sequences:

$$\begin{cases}
0 \to (\mathbb{Z}/2)^{2k+2} \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to 0, & q = 6k+2,\ q \geqslant 8 \\
0 \to (\mathbb{Z}/2)^{2k} \oplus \mathbb{Z}/3 \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 6k+1,\ q \geqslant 7 \\
0 \to (\mathbb{Z}/2)^{2k} \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to 0, & q = 6k+6, \\
0 \to (\mathbb{Z}/2)^{2k+4} \oplus \mathbb{Z}/3 \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 6k+5, \\
0 \to (\mathbb{Z}/2)^{2k} \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to 0, & q = 6k+4, \\
0 \to (\mathbb{Z}/2)^{2k+2} \oplus \mathbb{Z}/3 \to \mathrm{H}_q(\Gamma; \mathbb{Z}) \to 0, & q = 6k+3, \\
0 \to \mathbb{Z} \oplus (\mathbb{Z}/2)^2 \to \mathrm{H}_2(\Gamma; \mathbb{Z}) \to \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2 \to 0, & \\
0 \to \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2 \to \mathrm{H}_1(\Gamma; \mathbb{Z}) \to \mathbb{Z} \to 0. &
\end{cases}$$

Summarizing, we obtain:

$$\mathrm{H}_q(\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2, & q = 1,\\ \mathbb{Z} \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^2, & q = 2,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+2}, & q = 6k+3,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k+4,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+4}, & q = 6k+5,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+1}, & q = 6k+6,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k}, & q = 6k+1,\ q \geqslant 7,\\ \mathbb{Z}/3 \oplus (\mathbb{Z}/2)^{2k+3}, & q = 6k+2,\ q \geqslant 8. \end{cases}$$

We easily see these results except for the ambiguity in the 3-torsion of the short exact sequence for $\mathrm{H}_2(\Gamma;\mathbb{Z})$ and in the 2-torsion for all even degrees. To resolve it, we compute

$$\mathrm{H}_q(\Gamma;\mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^3, & q = 1,\\ (\mathbb{Z}/2)^5, & q = 2,\\ (\mathbb{Z}/2)^{4k+5}, & q = 6k+3,\\ (\mathbb{Z}/2)^{4k+3}, & q = 6k+4,\\ (\mathbb{Z}/2)^{4k+5}, & q = 6k+5,\\ (\mathbb{Z}/2)^{4k+5}, & q = 6k+6,\\ (\mathbb{Z}/2)^{4k-2}, & q = 6k+1,\ q \geqslant 7\\ (\mathbb{Z}/2)^{4k+3}, & q = 6k+2,\ q \geqslant 8. \end{cases}$$

and also homology with $\mathbb{Z}/3$-coefficients.

***Remark* 22.** For $m = 6$, the check introduced in remark 20 takes the following form. The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{A}, \overline{R}, \overline{U}, \overline{W} : 2\overline{A} = 0, 3\overline{R} = 0\rangle$. The parabolic element $U$ gives the cycle generating $\mathrm{H}_1(\Gamma\backslash X)$, whilst the parabolic element $W$ generates a trivial loop in the quotient space. So it follows that $E^{\infty}_{0,1} \cong \mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$, generated by $\overline{W}, \overline{A}$ and $\overline{R}$. This is consistent with the computation above.

3.5. $m = 15$. We have $\mathcal{O}_{\mathbb{Q}[\sqrt{-15}]} = \mathbb{Z}[\omega]$ with $\omega := -\frac{1}{2} + \frac{1}{2}\sqrt{-15}$.

Writing $\Gamma := \mathrm{PSL}_2(\mathbb{Z}[\omega])$ and

$$A := \pm\begin{pmatrix} & -1\\ 1 & \end{pmatrix}, \quad S := \pm\begin{pmatrix} & -1\\ 1 & 1\end{pmatrix}, \quad C := \pm\begin{pmatrix} 4 & -1-2\omega\\ 1+2\omega & 4\end{pmatrix},$$

$$T := \pm\begin{pmatrix} -3+\omega & -3-2\omega\\ -1-2\omega & 4\end{pmatrix}, \quad U := \pm\begin{pmatrix} 1 & 1+\omega\\ & 1\end{pmatrix},$$

$$V := \pm\begin{pmatrix} -1-2\omega & 3-\omega\\ 4 & 3+2\omega\end{pmatrix}, \quad W := \pm\begin{pmatrix} -1-2\omega & 4\\ 4+\omega & -1+2\omega\end{pmatrix},$$

we have the identifications $U^{-1}A \cdot (o,c) = (o',c')$, $T \cdot (a,b') = (a',b)$, $W \cdot (s,b') = (s,b)$, and $V^{-1} \cdot (s,a) = (s,a')$.

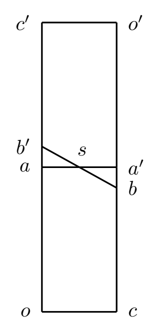



**Figure 5:** The fundamental domain for $m = 15$

There is no identification between the edges $(b,c)$ and $(b',c')$, nor between the edges $(a,o)$ and $(a',o')$. Thus the quotient by the $\Gamma$-action is homeomorphic to the sum of a Möbius band and a 2-sphere, with a disk amalgamated. There are five orbits of vertices, labelled $o, a, b, c, s$, with stabilizers

$$\begin{array}{ccccccc} \Gamma_o & = & \Gamma_a & = & \langle A \mid A^2 = 1\rangle & \cong & \mathbb{Z}/2,\\ \Gamma_c & = & \Gamma_b & = & \langle S \mid S^3 = 1\rangle & \cong & \mathbb{Z}/3,\\ & & \Gamma_s & = & \langle V, W \mid VW = WV\rangle & \cong & \mathbb{Z}^2. \end{array}$$

There are eight orbits of edges, labelled $(o,a)$, $(o',a')$,$(a,s)$, $(a,b')$, $(b,s)$, $(b,c)$, $(b',c')$ and $(o,c)$, amongst whose stabilizers only

$$\begin{array}{lclcccccc}
\Gamma_{(o,a)} & = & \langle A \mid A^2 = 1 \rangle & = & \Gamma_o & = & \Gamma_a & \cong & \mathbb{Z}/2, \\
\Gamma_{(o',a')} & = & \langle V^{-1}AV \mid (V^{-1}AV)^2 = 1 \rangle & = & \Gamma_{o'} & = & \Gamma_{a'} & \cong & \mathbb{Z}/2, \\
\Gamma_{(b,c)} & = & \langle S \mid S^3 = 1 \rangle & = & \Gamma_b & = & \Gamma_c & \cong & \mathbb{Z}/3, \\
\Gamma_{(b',c')} & = & \langle U^{-1}ASA^{-1}U \mid (U^{-1}ASA^{-1}U)^3 = 1 \rangle & = & \Gamma_{b'} & = & \Gamma_{c'} & \cong & \mathbb{Z}/3
\end{array}$$

are nontrivial; and four orbits of faces with trivial stabilizers. The above data gives the $\Gamma$-equivariant Euler characteristic of $X$, in accordance with remark 17:

$$\chi_\Gamma(X) = 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} - 4 - 2 \cdot \frac{1}{2} - 2 \cdot \frac{1}{3} + 4 = 0.$$

3.5.1. *Zeroth row of the $E^1$-term.*
We obtain in the columns $p = 0, 1, 2$:

$$\mathbb{Z}^5 \xleftarrow{d^1_{1,0}} \mathbb{Z}^8 \xleftarrow{d^1_{2,0}} \mathbb{Z}^4$$

where 1 is the only occurring elementary divisor of the differential matrices, with multiplicity four for $d^1_{1,0}$, and multiplicity three for $d^1_{2,0}$. The homology of this sequence is generated by the cycle $(o,a) + (a,b') + (b',c') + (c',o')$ in degree one and by the cycle $(a,s,b') - (a',s,b)$ in degree two.

3.5.2. *Even rows of the $E^1$-term.*
The even rows are the zero map to $E^1_{0,2} \cong \mathbb{Z}$, and to $E^1_{0,q} = 0$ for $q \geqslant 4$.

3.5.3. *Odd rows of the $E^1$-term.*
The maps

$$(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \xleftarrow{d^1_{1,q}} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$$

for $q \geqslant 3$, and

$$\mathbb{Z}^2 \oplus (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2 \xleftarrow{d^1_{1,1}} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/3)^2$$

are on the 2-primary part induced by the identity maps $\Gamma_{(o,a)} = \Gamma_o = \Gamma_a$ and $\Gamma_{(o',a')} = \Gamma_{o'} = \Gamma_{a'}$. So, we obtain the following rank 1 matrix for the 2-primary part:

$$(d^1_{1,q})_{(2)} = \begin{array}{c|cc} & (o,a) & (o',a') \\ \hline a & -1 & -1 \\ o & 1 & 1 \end{array}.$$

On the 3-primary part, they are induced by the identity maps $\Gamma_{(b,c)} = \Gamma_b = \Gamma_c$ and $\Gamma_{(b',c')} = \Gamma_{b'} = \Gamma_{c'}$. So, we obtain the following rank 1 matrix for the 3-primary part:

$$(d^1_{1,q})_{(3)} = \begin{array}{c|cc} & (b,c) & (b',c') \\ \hline b & -1 & -1 \\ c & 1 & 1 \end{array}.$$

3.5.4. *The $E^2$-page for $m = 15$.*
In the rows with $q \geqslant 2$, the $E^2$-page is concentrated in the columns $p = 0$ and $p = 1$:

$$\begin{array}{cc|cc}
q \geqslant 4 & \text{even} & 0 & 0 \\
q \geqslant 3 & \text{odd} & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 \\
q = 2 & & \mathbb{Z} & 0
\end{array}$$

Its lowest two rows are concentrated in the columns $p = 0, 1, 2$:

$$\begin{tikzcd}
q = 1 & \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3 & \mathbb{Z}/2 \oplus \mathbb{Z}/3 & 0 \\
q = 0 & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \arrow[ull, "d^2_{2,0}"']
\end{tikzcd}$$

3.5.5. *The $E^3 = E^\infty$-term.* For the calculation of the $d^2$-differential, we have

$$\begin{aligned}\delta\left((a,s,b') - (a',s,b)\right) &= (a,s) + (s,b') + (b',a) - (a',s) - (s,b) - (b,a')\\ &= (a,s) + W^{-1}\cdot(s,b) + (b',a) - V^{-1}\cdot(a,s) - (s,b) - T\cdot(b',a),\end{aligned}$$

$$\begin{aligned}(1\otimes\delta)(1\otimes_{(a,s,b')-(a',s,b)} 1) &= -(V^{-1}-1)\otimes_{(a,s)} 1 + (W^{-1}-1)\otimes_{(s,b)} 1 - (T-1)\otimes_{(b',a)} 1\\ &= (d_\Theta\otimes 1)\left(-(1,V^{-1})\otimes_{(a,s)} 1 + (1,W^{-1})\otimes_{(s,b)} 1 - (1,T)\otimes_{(b',a)} 1\right)\\ &= (d_\Theta\otimes 1)\left(-[V^{-1}]\otimes_{(a,s)} 1 + [W^{-1}]\otimes_{(s,b)} 1 - [T]\otimes_{(b',a)} 1\right).\end{aligned}$$

We then get

$$1\otimes\delta\left(-[V^{-1}]\otimes_{(a,s)} 1 + [W^{-1}]\otimes_{(s,b)} 1 - [T]\otimes_{(b',a)} 1\right) = [V^{-1}]\otimes_a 1 - [V^{-1}]\otimes_s 1 + [W^{-1}]\otimes_b 1 - [W^{-1}]\otimes_s 1 + [T]\otimes_{b'} 1 - [T]\otimes_a 1.$$

As the generators of the torsion-free part of $E^2_{0,1} \cong \mathbb{Z}^2 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/3$ are represented by $-[V^{-1}]\otimes_s 1$ and $-[W^{-1}]\otimes_s 1$, we see that the above computed element of $E^0_{0,1}$ represents an element $\nu \in E^2_{0,1}$ of infinite order with the following property: There is no element $\eta \in E^2_{0,1}$ with $k\eta = \nu$ for an integer $k > 1$. So, $E^3_{0,1} \cong \mathbb{Z} \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/2$ and $E^3_{2,0} = 0$.

3.5.6. *The short exact sequences.*
We thus obtain for integral homology the following short exact sequences:

$$\begin{cases} 0 \to \mathbb{Z}/2\oplus\mathbb{Z}/3 \to \mathrm{H}_q(\Gamma;\mathbb{Z}) \to 0, & q \geqslant 3,\\ 0 \to \mathbb{Z} \to \mathrm{H}_2(\Gamma;\mathbb{Z}) \to \mathbb{Z}/2\oplus\mathbb{Z}/3 \to 0, & \\ 0 \to \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3 \to \mathrm{H}_1(\Gamma;\mathbb{Z}) \to \mathbb{Z} \to 0. & \end{cases}$$

We obtain:

$$\mathrm{H}_q(\Gamma;\mathbb{Z}) \cong \begin{cases} \mathbb{Z}^2\oplus\mathbb{Z}/3\oplus\mathbb{Z}/2, & q = 1,\\ \mathbb{Z}\oplus\mathbb{Z}/3\oplus\mathbb{Z}/2, & q = 2,\\ \mathbb{Z}/3\oplus\mathbb{Z}/2, & q \geqslant 3.\end{cases}$$

We easily see these results except for the ambiguity in the 2-torsion and 3-torsion of the short exact sequence for $\mathrm{H}_2(\Gamma;\mathbb{Z})$. To resolve it, we compute homology with $\mathbb{Z}/2$- and $\mathbb{Z}/3$-coefficients,

$$\mathrm{H}_q(\Gamma;\mathbb{Z}/2) \cong \begin{cases} (\mathbb{Z}/2)^3, & q = 1 \text{ or } 2,\\ (\mathbb{Z}/2)^2, & q \geqslant 3.\end{cases} \qquad \mathrm{H}_q(\Gamma;\mathbb{Z}/3) \cong \begin{cases} (\mathbb{Z}/3)^3, & q = 1 \text{ or } 2,\\ (\mathbb{Z}/3)^2, & q \geqslant 3.\end{cases}$$

and then use the Universal Coefficient Theorem to compare.

***Remark* 23.** For $m = 15$, the check introduced in remark 20 takes the following form. The abelianization is $\Gamma^{\mathrm{ab}} \cong \langle \overline{AS}, \overline{C}, \overline{U} : 6\overline{AS} = 0\rangle$. The elements of infinite order $U$ and $C^{-1}$ give the same cycle, which generates $\mathrm{H}_1(\Gamma\backslash X)$. But the element $U^{-1}C^{-1}$ also has infinite order, and generates a trivial loop in the quotient space.
So it follows that $E^\infty_{0,1} \cong \mathbb{Z}\oplus\mathbb{Z}/2\oplus\mathbb{Z}/3$, generated by $\overline{U^{-1}C^{-1}}$ and $\overline{AS}$. This is consistent with the computation above.

## 4. Appendix: The equivariant retraction

In this section, we give Flöge's proof of the existence of a retraction $\rho$ from $\widehat{\mathcal{H}}$ to the cell complex $X^\bullet$. We do not show the fact that $\rho$ is $\Gamma$-equivariant, which can be observed since the fibers of $\rho$ are geodesic arcs.

**Theorem 24** ([9, theorem 6.6])**.** *$X$ is a retract of $\widehat{\mathcal{H}}$, i. e. there is a continuous map $\rho : \widehat{\mathcal{H}} \to X$ such that $\rho(p) = p$ for all $p \in X$.*

The map $\rho$ is first defined as the orthogonal projection $\pi$ from $\widehat{B}$ to $\partial\widehat{B}$, and is then continued to the whole of $\widehat{\mathcal{H}}$ by $\Gamma$. Bianchi [4] has shown that a nearly strict fundamental domain for the action of $\Gamma$ on $\mathcal{H}$ can be chosen in the form of a Euclidean vertical column $D$ inside $B$. Define

$$\widehat{D} := \{(z,r) \in \widehat{B} \quad | \quad 0 \leqslant \mathrm{Re}(z) \leqslant 1, \quad 0 \leqslant \mathrm{Im}(z) \leqslant \sqrt{m}\},$$

and denote by $S$ the set of singular points in $\widehat{D}$. Finally, $D := \widehat{D} - S$.

***Remark* 25** ([9], $D$ is $\Gamma$-normal)**.** For every $p \in \mathcal{H}$, there exists a neighborhood $U$ of $p$ in $\mathcal{H}$ such that there are at most finitely many $g \in \Gamma$ with $gD \cap U \neq \emptyset$.

We will use the following lemmas to prove the theorem 24.

**Lemma 26** ([9, lemma 6.5])**.** *For any subset $A \subset D$ which is closed in $\mathcal{H}$ and any $p \in \mathcal{H}$, there exists an open neighborhood $U_p$ of $p$ such that we have for all $g \in \Gamma$: $\quad gA \cap U_p \neq \emptyset$ if and only if $p \in gA$.*

*Proof.* By the above remark, there is a neighborhood $U$ of $p$ in $\mathcal{H}$ for which $\{g \in \Gamma \,|\, gD \cap U \neq \emptyset\}$ is finite. So especially its subset

$$\Gamma_o := \{g \in \Gamma \,|\, gA \cap U \neq \emptyset \quad \text{and} \quad p \notin gA\}$$

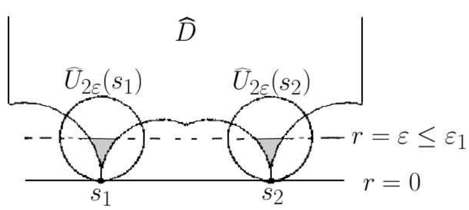
is finite. Therefore, $A$ being closed, $\bigcup\limits_{g \in \Gamma_o} gA$ is closed in $\mathcal{H}$. Thus $U_p := U - (\bigcup\limits_{g \in \Gamma_o} gA)$ is open in $\mathcal{H}$ and satisfies to the requested condition. $\square$

**Lemma 27** ([9, lemma 6.3])**.** *There is an $\varepsilon_0 > 0$ such that for all singular points $s, s' \in S$, for all $\varepsilon \leqslant \varepsilon_0$ and $g \in \Gamma$ we have the following statement: $\quad g\widehat{U}_\varepsilon(s) \cap \widehat{U}_\varepsilon(s') \neq \emptyset$ implies $gs = s'$.*

For class number two, as we obtain a fundamental domain for the action of $\Gamma$ on $\widehat{\mathcal{H}}$ (stricter than $\widehat{D}$) containing just one singular point, this lemma states only that $\Gamma$ acts discontinuously on $\widehat{\mathcal{H}}$ (with respect to its topology which is finer than the subset topology of $\mathbb{R}^3$); and we skip Flöge's proof which is useful for class number three or greater.

**Lemma 28** ([9, lemma 6.4])**.** *There exists an $\varepsilon_1 > 0$ with the following property:*
*If $\varepsilon \leqslant \varepsilon_1$ and $(z,r) \in \widehat{D}$ with $r < \varepsilon$, then there is an $s' \in S$ such that $(z,r) \in \widehat{U}_{2\varepsilon}(s')$.*

Flöge draws the following sketch of the situation in a vertical half-plane, which we reproduce here with his kind permission:




He gives only some hints on the proof, which we want to make slightly more explicit here.

*Sketch of proof.* We consider the Euclidean geometry of the upper-half space model for $\widehat{\mathcal{H}}$ and write coordinates in $\mathbb{C} \times \mathbb{R}^{\geqslant 0}$. Denote by $\varepsilon_1$ the "height of the lowest non-singular vertex", more precisely the minimum of the values $r > 0$ occuring as the real coordinate of the non-singular vertices $(z,r) \in \mathcal{H}$ of the fundamental domain $\rho(\widehat{D})$ for $\Gamma$. Then $\{(z,r) \in \widehat{D} \,|\, r < \varepsilon_1\}$ consists of one connected component for each singular point $s' \in S$. We will denote by $\widehat{D}_{s'}$ the connected component containing $s'$. Now fix $s' \in S$. There are finitely many hemispheres limiting $\widehat{D}$ from below and touching $s'$. We will consider the situation in a vertical half-plane containing $s'$. The most critical vertical half-planes for our assertion contain the intersection arc of two such hemispheres, because the other vertical

half-planes contain circle segments of $\partial\widehat{D}$ of greater radius. The intersection of two non-identical Euclidean 2-spheres which have more than one point in common, is a circle with center on the line segment connecting the two 2-sphere centers. Thus the intersection of the two hemispheres mentioned above is a semicircle with center in the plane $r = 0$ . Denote by $\zeta$ the radius of this semicircle. Then $\varepsilon_1 \leqslant \zeta$, because an edge of our fundamental domain, connecting $s'$ with a non-singular vertex, lies on this semicircle. Now it is easy to see that $\widehat{D}_{s'}$ is a subset of the truncated cone obtained as the convex envelope of $s'$ and the horizontal disk with radius $\zeta$ and center $(s', \zeta)$. We conclude that for all $\varepsilon < \varepsilon_1$, $\varepsilon > 0$, the set $\{(z, r) \in \widehat{D}_{s'} \,|\, r < \varepsilon\}$ is a subset of the horoball $\widehat{U}_{2\varepsilon}(s')$. So we have seen that $\varepsilon_1$ has the property claimed in the lemma. $\square$

*Proof of theorem 24.* For any $(z, r) \in \widehat{D}$ there is a unique $r_z \geqslant 0$ such that $(z, r_z) \in \widehat{D} \cap \partial\widehat{B} =: \widehat{G}$, in fact $r_z = \min\,\{r' : (z, r') \in \widehat{D}\}$. We can thus define the map $\pi : \widehat{D} \to \widehat{G}$ by $\pi(z, r) := (z, r_z)$. The map $\pi$ is continuous with respect to the subset topology of $\mathbb{R}^3$, and by [9, corollary 5.10] also with respect to the topology of $\widehat{\mathcal{H}}$. Furthermore, we have $\pi(p) = p$ for all $p \in \widehat{G}$. We now extend $\pi$ to a map $\rho : \widehat{\mathcal{H}} \to X$ as follows. Because of $\left\{\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right) : b \in R\right\} \cdot \widehat{D} = \widehat{G}$, we find for any $p \in \widehat{\mathcal{H}}$ a $\gamma \in \Gamma$ such that $\gamma(p) \in \widehat{D}$. We set $\rho(p) := \gamma^{-1} \circ \pi \circ \gamma(p)$. In order to show that this makes sense, we have to show that $p \in \gamma^{-1}\widehat{D} \cap \xi^{-1}\widehat{D}$ implies $\gamma^{-1} \circ \pi \circ \gamma(p) = \xi^{-1} \circ \pi \circ \xi(p)$, where $\gamma, \xi \in \Gamma$. We have $\xi(p) \in \xi\gamma^{-1}\widehat{D} \cap \widehat{D}$, then $\gamma\xi^{-1}(\xi(p)) = \gamma(p) \in \widehat{D} \cap \gamma\xi^{-1}\widehat{D}$, and either $\xi(p), \gamma(p)$ are both from $\widehat{G}$, or both from $\widehat{D} \cap B^\circ$. In the first case, it immediately follows that $\gamma^{-1} \circ \pi \circ \gamma(p) = \xi^{-1} \circ \pi \circ \xi(p) = p$, and $\xi^{-1} \circ \xi(p) = p$. In the second case, we have by [9, lemma 3.4] that if $\gamma\xi^{-1} = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, the entry $c$ must vanish. So $\gamma\xi^{-1}$ is the product $\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & db \\ 0 & 1.\end{smallmatrix}\right)$. Both of the latter two matrices commute with $\pi$ since any such element $\zeta$ satisfies $\zeta(\partial\widehat{B}) = \partial\widehat{B}$, and $\zeta$ maps vertical half-lines to vertical half-lines.
So we have $(\gamma\xi^{-1} \circ \pi \circ \xi\gamma^{-1})p' = \pi p'$ for all $p' \in \widehat{D}$ with $\xi\gamma^{-1}p' \in \widehat{D}$, and then it follows that

$$\xi^{-1} \circ \pi \circ \xi(p) = \gamma \in \gamma(\xi^{-1} \circ \pi \circ \xi)\gamma^{-1}\gamma(p) = \gamma^{-1} \circ \pi \circ \gamma(p) = \gamma^{-1} \circ \pi \circ \gamma(p).$$

Thus, $\rho$ is well-defined. Furthermore, $\pi(p) = p$ for all $p \in \widehat{G}$ implies $\rho(p) = p$ for all $p \in X$. It remains to show that $\rho$ is continuous at any $p \in \widehat{\mathcal{H}}$.
1st case. In the case $p \in \mathcal{H}$, by lemma 26, $p$ has an open neighborhood $U_p$ such that: for any $\gamma \in \Gamma$, we have $\gamma U_p \cap D \neq \emptyset \iff \gamma(p) \in D$. Furthermore, the set $\{\gamma \in \Gamma : \gamma(p) \in D\}$ is finite [9, remark 3.6], say $\gamma_1, \ldots, \gamma_n$. Let now $V$ be an open neighborhood of $\rho(p)$. Because of the continuity of all $\gamma_i, \gamma_i^{-1}$ and the continuity of $\pi : \widehat{D} \to \widehat{G}$, there exist neighborhoods $U_i$ of $p$ such that $\gamma_i^{-1} \circ \pi \circ \gamma_i(U_i) \subset V$. Note that for all $\gamma_i$ we have $\gamma_i^{-1} \circ \pi \circ \gamma_i(p) = \rho(p)$. Setting $U := U_p \cap (\bigcap_{i=1}^n U_i)$, we have $\rho(U) \subset V$, i. e. $\rho$ is continuous at the point $p$.
2nd case. In the case $p \in \widehat{\mathcal{H}} \cap \mathbb{C}$, let $\epsilon_0, \epsilon_1$ and $\epsilon_s$ for $s \in S$ be positive real numbers as in lemma 27, lemma 28 and [9, lemma 5.9]; and let $\epsilon > 0$ be less than the minimum of $\frac{\epsilon_0}{2}, \epsilon_1, \epsilon_s$ for $s \in S$. Because of $\left\{\left(\begin{smallmatrix} 1 & b \\ & 1\end{smallmatrix}\right) : b \in R\right\} \cdot \widehat{D} = \widehat{G}$, there exist $s \in S$, $\xi = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ such that $\xi s = p$ and by [9, remark 5.5(a)], we have $\xi\widehat{U}_\epsilon(s) = \widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)$. Let us now show that $\rho(\widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)) \subset \widehat{U}_{2\epsilon}(p)$. Let $p' \in \widehat{U}_{\frac{\epsilon}{|cs-d|^2}}(p)$, and let $\gamma \in \Gamma$ with $\gamma p' \in \widehat{D}$. Then $\rho(p') = \gamma^{-1} \circ \pi \circ \gamma(p')$. By [9, remark 5.5(b)], applied to $s$ and $\gamma\xi$ it follows that $\gamma p' = \gamma\xi(\xi^{-1}p') \in \widehat{U}_\epsilon(\gamma\xi s) = \widehat{U}_\epsilon(\gamma p)$, and by [9, remark 5.6] all conditions of lemma 28 are satisfied. So there is an $s' \in S$ such that $\gamma p' \in \widehat{U}_{2\epsilon}(s')$. This means that $\gamma\xi(\widehat{U}_{2\epsilon}(s)) \cap \widehat{U}_{2\epsilon}(s') \neq \emptyset$, and by lemma 27 it follows that $s' = \gamma\xi s = \gamma p$. Let us now consider $\gamma p'$ again.
Since $\gamma p' \in \widehat{U}_\epsilon(\gamma p)) = \widehat{U}_\epsilon(s') = U_\epsilon(s')$ and $\pi(U_\epsilon(s')) \subset U_\epsilon(s')$; and by [9, lemma 5.9] we have $U_\epsilon(s') \cap \widehat{B} \subset \widehat{U}_{2\epsilon}(s')$. So $\pi \circ \gamma p' \in \widehat{U}_{2\epsilon}(s')$. By [9, remark 5.5(b)] it finally follows that

$$\rho(p') = \gamma^{-1} \circ \pi \circ \gamma p' \in \gamma^{-1}\widehat{U}_{2\epsilon}(s') \subset \widehat{U}_{2\epsilon}(\gamma^{-1}s') = \widehat{U}_{2\epsilon}(p),$$

and we are done. $\square$

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